RESEARCH STATEMENT

CHUL-HEE LEE

1. Introduction

The theory of modular functions and that of q-hypergeometric series are both old and have been studied for a long time. Modular functions and modular forms arose out of the study of elliptic functions, pioneered by great mathematicians such as Abel, Jacobi, Weierstrass and Riemann and in particular out of the study of the symmetries of such functions. The notion of modularity is now pervasive in all of mathematics.

The study of hypergeometric functions goes back to Euler and Gauss. The development of the theory in the nineteenth century, notably by Kummer, Riemann, Schwarz and Klein, 'created a vision of the unity of mathematics' [Gra00] by relating it with linear differential equations, group theory, and non-Euclidean geometry. Since then, it has evolved into many different directions, a branch leading to the theory of q-hypergeometric series, also called basic hypergeometric series. It is full of beautiful formulas and has a vast range of applications, such as in orthogonal polynomials, combinatorics, Lie algebras and groups, and statistical mechanics.

Despite the long and rich history of the respective topics, our understanding of the connection between q-hypergeometric series and modular functions is far from being complete. In an attempt to provide a bridge between the two areas, several deep and intriguing conjectures have been formulated in recent years, and the central theme of my research has been to use ideas from representation theory and mathematical physics to resolve some of these conjecture and to thus advance our understanding of the interplay between q-hypergeometric series and modular functions.

The story of q-hypergeometric functions and their link to the modular world started well over a hundred years ago with the famous Rogers-Ramanujan identities which we will see shortly. Let us first take a look at an identity of Ramanujan, which is still a source of motivation and inspiration to many number theorists. In a letter of Ramanujan to Hardy, Ramanujan presents an identity :

$$\frac{1}{1+\cfrac{e^{-2\pi}}{1+\cfrac{e^{-4\pi}}{1+\cfrac{e^{-6\pi}}{\ddots}}}} = \left(\frac{\sqrt{5+\sqrt{5}}}{2} - \frac{\sqrt{5}+1}{2}\right) e^{2\pi/5}. \tag{1}$$

This identity represents a beautiful interaction between the theory of q-hypergeometric series and the theory of complex multiplication, which forms an integral part of the theory of modular functions. It can be derived from the Rogers-Ramanujan identities :

$$G(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}, \quad H(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty} \tag{2}$$

where $(a;q)_n = (1-a)(1-aq)(1-aq^2)\dots(1-aq^{n-1})$. The left-hand side in each of (2), given as a series, is an example of q-hypergeometric series. The right-hand side, represented by an infinite product, is an example of modular function. In general, there is no easy way to tell whether a given q-hypergeometric series has modularity.

The continued fraction in the left-hand side of (1) can be explained by a hypergeometric type functional equation satisfied by the Rogers-Ramanujan identities. The right-hand side is an example of singular moduli [GZ85], the values of modular functions at quadratic imaginary integers.

Date: February 26, 2019.

There is another interesting identity, involving the dilogarithm function [Kir95, Zag07], which is again due to the interaction between the two theories. One virtue of modular functions is that the behavior at the points where q is a root of unity can be nicely described. Both series $G(q)$ and $H(q)$ converge for $|q| < 1$ but thanks to modularity we have full control of the blow up of them as q approaches a root of unity. A modular function $f(q)$ behaves asymptotically as $\exp\left(\frac{r\pi^2}{t}\right)$ as $t \searrow 0$ for some rational number r, where $q = e^{-t}$.

For a q-hypergeometric function to be modular, the behavior of q-series as $t \searrow 0$ must match that of modular functions. For a positive real number $A > 0$, we have

$$\sum_{n=0}^{\infty} \frac{q^{\frac{A}{2}n^2+Bn}}{(q)_n} \sim \exp\left(\frac{L(x)}{t}\right)$$

where x is the unique positive solution of the equation

$$x = (1-x)^A \tag{3}$$

and L is the Rogers dilogarithm function. One can see that the requirement of modularity puts a certain constraint on the possible values of A since $L(x)$ must be a rational multiple of $L(1) = \pi^2/6$.

In the case of the Rogers-Ramanujan identities, the above asymptotics gives

$$G(q), H(q) \sim \exp\left(\frac{\pi^2}{15t}\right).$$

The number $\frac{\pi^2}{15}$ inside the exponential term is obtained from the identity

$$L\left(\frac{3-\sqrt{5}}{2}\right) = \frac{\pi^2}{15} = \frac{2}{5}L(1). \tag{4}$$

Interestingly, this $\frac{2}{5}$ predicts the appearance of 2 congruence classes modulo 5 on the infinite product side of (2). This number, with such a purely arithmetic meaning, counting congruence classes appearing in partition identities will be the effective central charge of a certain model in the context of conformal field theory (CFT). A CFT is a quantum field theory invariant under conformal transformations. In two dimensions, it has rich mathematical structures and the development of the theory has led to remarkable mathematical progress in many areas of mathematics and physics. The central charge of a CFT is one of the most important numbers governing the theory.

The fact that the central charge can be obtained from the evaluation of the Rogers dilogarithm function is related to the existence of integrable perturbations of the relevant CFT, whose precise mathematical formulation still seems to be missing. Roughly speaking, the modularity of (2) belongs to the world of CFTs and its representation as a q-hypergeometric series reflects the structures of their integrable perturbations. Thus the dilogarithm identity (4) can be viewed as a consequence of deep facts about CFTs.

Although this is a simple example, it already shows some ingredients central to our study : the algebraic equation (3) associated to A, the dilogarithm identity (4) involving the solution of the equation and its lifts to (2), which is a q-hypergeometric series with modularity. More generally, we want to find a positive definite symmetric matrix A for which there exists a corresponding dilogarithm identity and its lifts.

Nahm's conjecture. As indicated above, finding a criterion for a q-hypergeometric series to be a modular function is a widely-open problem in number theory. Nahm's conjecture addresses this question [Nah07, Zag07]. Motivated by the idea of thermodynamic Bethe Ansatz in mathematical physics and integrable perturbations of rational CFTs, Nahm considered r-fold q-hypergeometric series of the form

$$f_{A,B,C}(z) = \sum_{\mathbf{n}\in(\mathbb{Z}_{\geq 0})^r} \frac{q^{\frac{1}{2}\mathbf{n}^t A\mathbf{n}+B\cdot\mathbf{n}+C}}{(q)_{n_1}\cdots(q)_{n_r}} \tag{5}$$

where $q = e^{2\pi iz}$, $\mathbf{n} = (n_1, \ldots, n_r)$, A is a positive definite $r \times r$ symmetric matrix, and B a vector and C a scalar, all with entries in $\mathbb{Q}$. If $f_{A,B,C}$ is a modular function, then we call (A, B, C) a *modular triple* and A the matrix part of it.

He conjectured that modularity of these series is dictated by torsion elements of the associated Bloch group, an object in algebraic K-theory. In an attempt to characterize the matrix A in a modular triple, he considers the asymptotic behavior of $f_{A,B,C}$ when z approaches 0. This leads to a system of equations associated to the matrix $A = (a_{ij})$, given by

$$x_i = \prod_{j=1}^{r}(1-x_j)^{a_{ij}}, \quad i = 1, \dots, r. \tag{6}$$

This is a system of r equations of r variables $x_1, \dots, x_r$ and will be denoted by $\mathbf{x} = (1-\mathbf{x})^A$.

For a solution $\mathbf{x} = (x_1, \dots, x_r)$ of $\mathbf{x} = (1-\mathbf{x})^A$, we consider a formal sum $\xi_{\mathbf{x}} = [x_1] + \cdots + [x_r]$ in the group ring $\mathbb{Z}[F]$ of a number field F containing the solution $(x_1, \dots, x_r)$. This can be regarded as an element of the Bloch group $\mathcal{B}(F)$. Nahm's conjecture claims that there exists a modular triple (A, B, C) if and only if the element $\xi_{\mathbf{x}} \in \mathcal{B}(F)$ is a torsion element for any solution $x = (x_1, \dots, x_r) \in F^r$ of (6)

There are not many known results about Nahm's conjecture. Zagier completed the classification of modular triples when A is of rank 1 in [Zag07]. It says that only three matrices are allowed as a matrix part of a modular triple : $(\frac{1}{2}), (1)$ and (2). Vlasenko and Zwegers in [VZ11] found counterexamples to Nahm's conjecture. More specifically, they found some modular triples (A, B, C) such that not all of the solutions of $\mathbf{x} = (1-\mathbf{x})^A$ are torsion in the Bloch group. Thus we need to find a reformulation of the conjecture if we intend not to abandon the whole conjecture. Guided by some observations from [HL10], which was about the matrices of size 2 satisfying the condition in Nahm's conjecture, one result we obtained in this study is as follows :

Theorem 1. [Lee13a] *For a matrix $A = \mathcal{C}(X) \otimes \mathcal{C}(X')^{-1}$ where (X, X') is a pair of ADE diagrams and $\mathcal{C}(\cdot)$ denotes its Cartan matrix, every solution $\mathbf{x}$ of the equation $\mathbf{x} = (1-\mathbf{x})^A$ in a number field F gives rise to a torsion element of the Bloch group $\mathcal{B}(F)$.*

For its proof, we have used the known properties of Y-systems [Nak11, Kel13] and the dilogarithm function. Once we collect the necessary ingredients, the proof follows without much difficulty. The main point of this result was not on the new tools or techniques but rather on its meaning, or what it tries to tell. The fact that the matrix A associated to a pair of Dynkin diagrams satisfies the K-theoretic properties implied by Nahm's conjecture, suggests that there might be a certain link between the conjecture and Lie theory. That was a new direction that we took for further research afterwards.

Nahm's conjecture and representation theory. The mysterious appearance of the Cartan matrices poses several questions immediately. Does Nahm's conjecture hold in this case? In other words, is there a family of modular q-hypergeometric series associated to this special form of matrices? If it is so, do they have a representation theoretic origin? We have made some effort to put Theorem 1 into a bigger mathematical context, which seems to be found in representation theory of classical and quantum affine algebras.

To be more specific, let us introduce some notation. Let $\mathfrak{g}$ be a complex simple Lie algebra of rank r. Let I be the set of nodes of its Dynkin diagram and $\{\alpha_a\}_{a\in I}$ be the set of simple roots. Let $t_a = (\theta, \theta)/(\alpha_a, \alpha_a) \in \{1, 2, 3\}$ for each $a \in I$, where θ is the highest root and $(\cdot, \cdot)$ is the bilinear form on the weight lattice, induced from the Killing form. Let us fix a positive integer $k \geq 2$.

Let us define a positive definite matrix $K = (K_{ij})_{i,j\in H_k}$ with rational entries, where $H_k = \{(a, m) | a \in I, 1 \leq m \leq t_a k - 1\}$, as follows:

$$K_{(a,m),(b,n)} = \Big(\min(t_b m, t_a n) - \frac{mn}{k}\Big)(\alpha_a, \alpha_b). \tag{7}$$

Note that K becomes $\mathcal{C}(\mathfrak{g}) \otimes \mathcal{C}(A_{k-1})^{-1}$ when $\mathfrak{g}$ is simply-laced. We can consider a q-hypergeometric series associated to K of the form (5) with a choice of vector and scalar.

Of course, by focusing on this special family, we are losing some generality that Nahm's conjecture tries to achieve in the study of modular q-hypergeometric series. However, this loss is compensated by the fact that this family seems to possess very rich underlying algebraic structures and beautiful connections with other areas of mathematics. Despite its importance and elegance, the theory about this family has been much less developed than it deserves.

A large part of my research can be now summarized as a study of this family of q-hypergeometric series associated to K. In the next section, we will give a brief exposition of the following topics :

(1) **Q-systems and fusion rings**
(2) **linear recurrence relations in Q-systems and applications**

(3) **modular q-hypergeometric series in representation theory**.

In addition to this, we will also give some remarks on the ongoing work on Macdonald-Koornwinder polynomials and how it fits with the above story.

2. Results and problems

2.1. **Q-systems and fusion rings.** Let us consider the equation $\mathbf{x} = (1-\mathbf{x})^K$. A link between this and representation theory is provided by the fact that we can find solutions of $\mathbf{x} = (1-\mathbf{x})^K$ by solving the following system of equations in $\left(Q_m^{(a)}\right)_{a\in I, 0\le m\le k}$ over the field of complex numbers :

$$\left(Q_m^{(a)}\right)^2 = Q_{m-1}^{(a)}Q_{m+1}^{(a)} + \prod_{b:C_{ab}\neq 0}\prod_{j=0}^{-C_{ab}-1} Q^{(b)}_{\lfloor \frac{C_{ba}m-j}{C_{ab}}\rfloor}, \quad a\in I, 0\le m\le t_a k \tag{8}$$

where $Q_{-1}^{(a)} = Q_{t_a k+1}^{(a)} = 0$.

Interestingly, (8) has been studied in the representation theory community. The Kirillov-Reshetikhin (KR) modules form a special class of finite-dimensional representations of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$, which is a class of quantum groups. Here, $q\in\mathbb{C}^\times$ is not a root of unity. For each $a\in I$, $m\in\mathbb{Z}_{\ge 0}$, and $u\in\mathbb{C}^\times$, there is an associated KR module $W_m^{(a)}(u)$. By restriction we obtain a finite-dimensional $U_q(\mathfrak{g})$-module res $W_m^{(a)}(u)$. As its dependence on u disappears as $U_q(\mathfrak{g})$-module, we can simply write it as res $W_m^{(a)}$.

When we set $Q_m^{(a)}$ to be the character of $\operatorname{res} W_m^{(a)}$, then the family $(Q_m^{(a)})_{a\in I, m\in\mathbb{Z}_{\ge 0}}$ satisfies (8), now for any $m\in\mathbb{Z}_{\ge 0}$; we call it the (unrestricted) *Q-system.* This was first conjectured in [KR87] and proved by Nakajima [Nak03] in simply-laced types and by Hernandez [Her06] in general.

To understand the role of Q-systems in mathematical physics, let us make a little digression. The Bethe Ansatz is a very powerful technique in the study of exactly solvable models and integrable systems. In 1928, Heisenberg proposed a quantum mechanical model of magnets, which can be considered as a prototypical example of a large class of exactly solvable models. Bethe introduced his ansatz to find the spectrum of the Heisenberg Hamiltonian in 1931. The search for algebraic structures behind his solution and its extensions eventually led to the discovery of quantum groups.

Kirillov and Reshetikhin carried out pioneering work [KR87], bringing this tool from mathematical physics to the representation theory of Yangians and quantum affine algebras, which are important classes of quantum groups. In this work, they introduced the Q-system, associated to each simple Lie algebra $\mathfrak{g}$. As clarified in [HKO+99], the Q-system plays an important role in establishing the combinatorial completeness of Bethe states, which is incarnated as a formula, called the fermionic formula, for the decomposition of a tensor product of Kirillov-Reshetikhin modules under restriction. In general, it is a fundamental and notoriously difficult problem to establish the completeness of Bethe states for various kinds of exactly solvable models. The rigorous mathematical results established for representations of quantum affine algebras give valuable insight to understand the potential underlying mechanism of Bethe Ansatz for more broad range of problems studied in mathematical physics.

Let us go back to our main story. A conjecture formulated by Kirillov [Kir89] and Kuniba, Nakanishi, and Suzuki [KNS94] claims that one can solve (8) by appropriately specializing the characters of KR modules; note that (8) is a finite set of equations in finitely many variables in contrast to the unrestricted Q-system. In particular, it claims that the unique positive real solution of (8) is given by the quantum dimensions of KR modules. The conjecture has its origin in dilogarithm identities for CFTs and it had been open except for type A without any essential progress. In [Lee13b, Lee17], we managed to prove it for all classical types.

Theorem 2. *Let $\mathfrak{g}$ be of types A_r, B_r, C_r and D_r. Let $\mathcal{D}_m^{(a)} := Q_m^{(a)}(\frac{\rho}{k+h^\vee})$ be the quantum dimension of $W_m^{(a)}$. For each $a\in I$, the following properties hold :*

- *(i) $\mathcal{D}_m^{(a)} > 0$ for $0\le m\le t_a k$,*
- *(ii) $\mathcal{D}_{t_a k+1}^{(a)} = \mathcal{D}_{t_a k+2}^{(a)} = \cdots = \mathcal{D}_{t_a(k+h^\vee)-1}^{(a)} = 0$,*
- *(iii) $\mathcal{D}_{m-1}^{(a)} < \mathcal{D}_m^{(a)}$ for $1\le m\le \lfloor \frac{t_a k}{2}\rfloor$.*

Our new insight has been to introduce the fusion ring into the study of Q-systems. It not only solves an old open problem but also opens up a new area of research. In fact, we were able to construct solutions of Q-systems satisfying certain boundary conditions in the fusion ring of the category of integrable representations of the affine Lie algebra $\widehat{\mathfrak{g}}$. The fusion ring is an important object in many areas of mathematics and physics, including algebraic geometry, conformal field theory and topological quantum field theory [Bea96, BK01]. In relation to Nahm's conjecture, solving Q-systems with boundary conditions is roughly equivalent to solving equations associated to a pair of Cartan matrices appearing in the conjecture.

To explain the central idea of the proof, let us introduce two ring homomorphisms. There are many known results such as the decomposition of the KR modules over $U_q(\widehat{\mathfrak{g}})$ as $U_q(\mathfrak{g})$-modules concerning the surjective homomorphism $\mathcal{R}(U_q(\widehat{\mathfrak{g}})) \xrightarrow{\text{res}} \mathcal{R}(U_q(\mathfrak{g}))$. Here $\mathcal{R}(\cdot)$ denotes the Grothendieck ring of the category of finite-dimensional representations of relevant quantum group. Now consider the category of integrable representation of affine Lie algebra $\widehat{\mathfrak{g}}$ of level k equipped with fusion product. Its Grothendieck ring $\mathcal{R}_k(\mathfrak{g})$ is called the fusion ring and it is a free $\mathbb{Z}$-module of finite rank with a distinguished basis. In the study of $\mathcal{R}_k(\mathfrak{g})$, a surjective homomorphism $\beta_k : \mathcal{R}(U_q(\mathfrak{g})) \to \mathcal{R}_k(\mathfrak{g})$ plays an important role; see [Bea96, BK09].

It turns out that it is essential to study the composition $\mathcal{R}(U_q(\widehat{\mathfrak{g}})) \xrightarrow{\text{res}} \mathcal{R}(U_q(\mathfrak{g})) \xrightarrow{\beta_k} \mathcal{R}_k(\mathfrak{g})$ of these two homomorphisms, which had been studied separately and not been considered before in this way. We found that the conjecture of Kirillov and Kuniba-Nakanishi-Suzuki admits a fusion ring reformulation and it makes many aspects of the problem clear. Now what we need to know is $\beta_k(\operatorname{res} W_m^{(a)})$ for each (a, m). This approach and its partial progress led us to a proof of Theorem 2 and a set of new conjectures, which we can summarize as follows :

Conjecture 3. [Lee17] *For each $a \in I$, the following properties hold :*

(i) $\beta_k(\operatorname{res} W_m^{(a)})$ *is positive for* $0 \le m \le t_a k$,

(ii) $\beta_k(\operatorname{res} W_{t_a k+1}^{(a)}) = \beta_k(\operatorname{res} W_{t_a k+2}^{(a)}) = \cdots = \beta_k(\operatorname{res} W_{t_a(k+h^\vee)-1}^{(a)}) = 0$,

(iii) $\beta_k(\operatorname{res} W_{m+nt_a(k+h^\vee)}^{(a)}) = \sigma_a^n V_{k\hat{\tau}_a}^n \beta_k(\operatorname{res} W_m^{(a)})$ *for* $0 \le m \le t_a(k+h^\vee) - 1$ *and* $n \in \mathbb{Z}_{\ge 0}$.

Here $V_{k\hat{\tau}_a}$ *is a certain unit element in the fusion ring and* $\sigma_a \in \{\pm 1\}$.

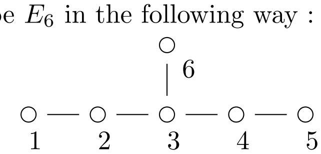
To put the periodicity part (iii) of Conjecture 3 in the right context, we need to consider linear recurrence relations in Q-systems; we will explain this in the next subsection in more detail. The key point here is the existence of linear recurrences implies the periodicity in Conjecture 3. Once this problem is settled, the resolution of Conjecture 3 is essentially reduced to knowing $\beta_k(\operatorname{res} W_m^{(a)})$ for $0 \le m \le t_a k$.

The positivity problem (i) of Conjecture 3 is still poorly understood and requires some new ideas. A Bethe ansatz-like physical combinatorics might be one of them. There exists a fermionic formula for $\mathcal{R}(U_q(\mathfrak{g}))$, which was one of the original motivation to study the KR modules. The reformulation of the problem now opens up the possibility of having the fermionic formula for the fusion ring $\mathcal{R}_k(\mathfrak{g})$ in an analogous way. To find the right combinatorial objects explaining the positivity is an open problem.

2.2. **linear recurrence relations in Q-systems and applications.** For $a \in I$ fixed, consider the sequence $\{Q_m^{(a)}\}_{m=0}^{\infty}$. The periodicity (iii) of Conjecture 3 partly motivated me to study linear recurrence relations in this sequence. In [Lee15], some structural properties of these recurrences were conjectured based on extensive computer experiments, which we later managed to prove :

Theorem 4. [Lee19a] *Assume that $\mathfrak{g}$ is a simple Lie algebra which is not of type E_7 or E_8. For each $a \in I$, $\{Q_m^{(a)}\}_{m=0}^{\infty}$ satisfies a linear recurrence relation with constant coefficients such that its characteristic equation is without multiple roots and the set of roots is invariant under the Weyl group.*

In addition to the applications to Conjecture 3, this linear recurrence relation has other interesting applications to representation theory of $U_q(\widehat{\mathfrak{g}})$. Let us illustrate it by giving a concrete example. Let us enumerate the nodes of the Dynkin diagram of type E_6 in the following way :



Conjecture 5. [HKO+99] *Let $\mathfrak{g}$ be a simple Lie algebra of type E_6. For any $m \in \mathbb{Z}_{\ge 0}$, we have*

$$\operatorname{res} W_m^{(3)} = \sum_{\substack{j_1+2j_2+j_3+j_4 \le m \\ j_1,j_2,j_3,j_4 \in \mathbb{Z}_{\ge 0}}} p(j_1,j_2,j_3,j_4) L\left(j_1\lambda_1 + j_2\lambda_2 + j_3\lambda_3 + j_4\lambda_4\right), \tag{9}$$

where $p(j_1, j_2, j_3, j_4) = (j_4+1) \times \min(1 + j_3,\, 1 + m - j_1 - 2j_2 - j_3 - j_4)$ *and* $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (\omega_1+\omega_5, \omega_2+\omega_4, \omega_3, \omega_6)$*. Here* $L(\lambda)$ *denotes an irreducible highest weight representation of* $U_q(\mathfrak{g})$ *with highest weight* λ *and* ω_a *is the fundamental weight for each* $a \in I$*.*

Even though we have a way to describe res $W_m^{(3)}$ using the fermionic formula, showing that it coincides with the right-hand-side of (9) is a different problem. Having a such a formula is sometimes very useful to study KR modules. By applying Theorem 4 to this, we obtain

Theorem 6. [Lee19a] *Conjecture 5 holds if* (9) *is true for* $m = 0, \ldots, 6894$*.*

This reduces a proof of Conjecture 5 to a finite amount of computation. In general, when the decomposition of Kirillov-Reshetikhin modules has an irreducible summand with multiplicity greater than 1, a family of formula similar to (9) has remained conjectural except for type G_2. Now we have a tool to investigate further the remaining cases in exceptional types. Recently, we obtained a proof of such a polyhedral formula in type F_4 [Lee19b]. As another application of Theorem 4, we obtain :

Theorem 7. [Lee19a] *Let* $\mathfrak{g}$ *be a simple Lie algebra. Define* $q_a : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ *to be* $q_a(m) = \dim W_m^{(a)}$ *for each* $a \in I$*. Then* q_a *is a quasipolynomial of period dividing* t_a*, of positive degree, and the positive constant leading coefficient. It satisfies the following reciprocity :*

$$q_a(-m) = (-1)^{e_a} q_a(m - t_a h^\vee), \qquad m \in \mathbb{Z}, \tag{10}$$

where e_a *is the degree of* q_a*.*

The properties of this quasipolynomial are consistent with the ones appearing in Ehrhart theory of convex polytopes. Ehrhart theory is the study of lattice points inside rational polytopes. A rational polytope $\mathcal{P} \subset \mathbb{R}^n$ is the convex hull of finitely many points in $\mathbb{Q}^n$. The lattice point enumerator of $\mathcal{P}$ is defined to be

$$L_{\mathcal{P}}(m) := \#(m\mathcal{P} \cap \mathbb{Z}^n),\ m \in \mathbb{Z}_{\geq 0}$$

where, $m\mathcal{P}$ is the m-th dilate of $\mathcal{P}$. The fundamental theorem of Ehrhart states that $L_{\mathcal{P}}(m)$ is a quasipolynomial of degree $\dim \mathcal{P}$. For $L_{\mathcal{P}}(m)$, a similar formula to (10) is called Ehrhart-Macdonald reciprocity.

Conjecture 8. [Lee19a] *Let* $\mathfrak{g}$ *be a simple Lie algebra. For each* $a \in I$*, there exists a rational polytope* $\mathcal{P}^{(a)}$ *of dimension* e_a *such that the set of lattice points of* $m\mathcal{P}^{(a)}$ *has an affine crystal structure, isomorphic to the KR crystal associated with* $W_m^{(a)}$*.*

We expect that it may have some roles in understanding the crystal bases of Kirillov-Reshetikhin modules, whose existence [FOS09] is still conjectural in general.

There is another interesting application [Lee18] of Theorem 4.

2.3. **modular q-hypergeometric series in representation theory.** Now we turn our attention to modular q-hypergeometric series associated to K, given in (7). A very mysterious fact is that both the finite-dimensional representations of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ and the infinite-dimensional representations of the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ are involved here again. The unique positive real solution of equation $\mathbf{x} = (1-\mathbf{x})^K$, appearing in Nahm's conjecture, can be obtained from Conjecture 3 and in this way, the story is related to the KR-modules of $U_q(\widehat{\mathfrak{g}})$, which are finite-dimensional.

Let Λ be a dominant integral affine weight and $L(\Lambda)$ be an irreducible highest weight representation of $\widehat{\mathfrak{g}}$ with highest weight Λ. For each weight λ of $L(\Lambda)$, the string function c_λ^Λ is defined by

$$c_\lambda^\Lambda = e^{-m_{\Lambda,\lambda}\delta} \sum_{n=-\infty}^{\infty} \mathrm{mult}_\Lambda(\lambda - n\delta) e^{-n\delta}$$

where $m_{\Lambda,\lambda}$ is a certain rational number and $\mathrm{mult}_\Lambda(\mu)$ is the multiplicity of the weight space of $L(\Lambda)$ with weight μ. Due to the work of Kac and Peterson [KP84] it is known to be a modular form of weight $-r/2$ if we set $q = e^{-\delta}$, but its explicit expression is not known in general. A long-standing conjecture related to our original question is the following :

Conjecture 9. [KNS93] *Let* $k \geq 2$ *be an integer and* $A = K$*. Let* λ *be a weight of* $\widehat{\mathfrak{g}}$*-module* $L(k\Lambda_0)$*. Up to a rational multiple of* $q = e^{2\pi i \tau}$*,*

$$\eta(\tau)^r \cdot c_\lambda^{k\Lambda_0}(\tau) = \sum_{\mathbf{n} \in (\mathbb{Z}_{\geq 0})^{|H_k|}} \frac{q^{\frac{1}{2}\mathbf{n}^t A \mathbf{n}}}{(q)_{\mathbf{n}}}, \tag{11}$$

where the sum is over $\mathbf{n} = (N_m^{(a)})_{a\in I, 1\le m\le t_a k-1} \in (\mathbb{Z}_{\ge 0})^{|H_k|}$ *such that*

$$\sum_{(a,m)\in H_k} m N_m^{(a)} \alpha_a \equiv \overline{\lambda} \mod kQ^\vee.$$

Here, η *is the Dedekind eta function and* $Q^\vee = \oplus \mathbb{Z}\alpha_a^\vee$, *where* $\alpha_a^\vee = t_a \alpha_a$.

Due to the known modularity of string functions, it will give an example of modular q-hypergeometric series associated to K, once (11) is established. Note that in (11), the vector part B is zero, comparing it with (5). Thus, this cover only some special cases of q-hypergeometric series associated to K and it is also necessary to find all triples (A, B, C) with $A = K$ its matrix part and establish the modularity of the corresponding q-hypergeometric series.

In [CNV10], the authors considered the trace of quantum monodromy operator associated to a pair of Dynkin diagrams and explained how one can produce q-hypergeometric series involving a tensor product of two Cartan matrices based on the idea of wall-crossing invariance. The quantum dilogarithm is a central object here and it would be an interesting project to find a way to link it directly to the above conjectures. The idea of wall-crossing invariants has not been discussed and studied much by number theorists and I believe much work should be done following this idea, which is what I intend to do in the future.

2.4. **ongoing work on Macdonald-Koorwinder polynomials.** In [BW15, GOW16, RW], Warnaar and his collaborators obtained a new family of Rogers-Ramanujan identities. Here, the sum side of an identity is expressed in terms of specialization of ordinary Hall-Littlewood polynomials and the other side is given as a product of theta functions. Let us briefly describe some key steps to obtain such an identity :

(1) Write the character of $\widehat{\mathfrak{g}}$-module $L(m\Lambda_0)$ as a sum of modified Hall-Littlewood polynomials $P'_\lambda(x;t)$, which is called a combinatorial character formula. At this stage, both sides involve variables $x_1, \ldots, x_n, t$.
(2) Specialize $x_1, \ldots, x_n, t$ appropriately so that we get an expression involving only q. Then we have an identity of the form "character" = "sum of modified Hall-Littlewood polynomials", where both sides are functions in q.
(3) Once we express the character as a product of theta functions and modified Hall-Littlewood polynomials as ordinary Hall-Littlewood polynomials, we obtain a new Rogers-Ramanujan identity.

A crucial new ingredient found in [RW] is that the first step in the above can be established as a suitable limit of bounded Littlewood identities for Macdonald polynomials. One of bounded Littlewood identities obtained by Rains and Warnaar is, for example,

$$\sum_{\substack{\lambda \\ \lambda_1 \le m}} b^{\mathrm{el}}_{\lambda;m}(q,t) P_\lambda(x;q,t) = (x_1\cdots x_n)^{\frac{m}{2}}\, P^{(\mathrm{B}_n,\mathrm{B}_n)}_{(\frac{m}{2})^n}(x;q,t,t) \tag{12}$$

where λ is a partition of size n, $x = (x_1, \ldots, x_n)$, m is a nonnegative integer,

$$b^{\mathrm{el}}_{\lambda;m}(q,t) := \prod_{\substack{s\in\lambda \\ l'(s)\ \text{even}}} \frac{1-q^{m-a'(s)}t^{l'(s)}}{1-q^{m-a'(s)-1}t^{l'(s)+1}} \prod_{\substack{s\in\lambda \\ l(s)\ \text{even}}} \frac{1-q^{a(s)}t^{l(s)}}{1-q^{a(s)+1}t^{l(s)}},$$

and a, a', l, l' are various statistics related to partitions. The polynomial $P_\lambda(x;q,t)$ is a Macdonald polynomial of type A and $P^{(\mathrm{B}_n,\mathrm{B}_n)}_{(\frac{m}{2})^n}(x;q,t,t)$ is a Macdonald polynomial attached to the pair of root system $(\mathrm{B}_n, \mathrm{B}_n)$. Equation (12) is a q,t-analogue of a known branching formula for Lie algebras of classical type.

One way to further explore the identities like (12) comes from the fact that a Macdonald polynomial of classical type is a specialization of the Koornwinder polynomials $K_\lambda(x;q,t;t_0,t_1,t_2,t_3)$, which are symmetric polynomials with six parameters. For example, $P^{(\mathrm{C}_n,\mathrm{B}_n)}_\lambda(x;q,t,t_2)$ is given by :

$$P^{(\mathrm{C}_n,\mathrm{B}_n)}_\lambda(x;q,t,t_2) = K_\lambda\big(x;q,t;q^{1/2},-q^{1/2},t_2^{1/2},-t_2^{1/2}\big).$$

In our collaboration with Warnaar, we have obtained new conjectural identities along this line, such as

$$P_{m^r}(x^{\pm};q,t) = \sum_{\lambda\subseteq m^r} f^{(m,n,r)}_{m^r-\lambda}(q,t;\pm q^{1/2},\pm t^{1/2})\, P^{(\mathrm{C}_n,\mathrm{B}_n)}_\lambda(x;q,t,t) \tag{13}$$

where f has a nice product expression. An in-depth investigation involving classical summation formulas in basic hypergeometric series and the Koornwinder integrals is still being carried out.

References

[Bea96] A. Beauville, *Conformal blocks, fusion rules and the Verlinde formula*, Israel Math. Conf. Proc., vol. 9, Bar-Ilan Univ., Ramat Gan, 1996, pp. 75–96.

[BK01] B. Bakalov and A.A. Kirillov, *Lectures on tensor categories and modular functors*, University Lecture Series, American Mathematical Society, 2001.

[BK09] A. Boysal and S. Kumar, *A conjectural presentation of fusion algebras*, Algebraic analysis and around, Adv. Stud. Pure Math., vol. 54, Math. Soc. Japan, Tokyo, 2009, pp. 95–107.

[BW15] N. Bartlett and S. O. Warnaar, *Hall-Littlewood polynomials and characters of affine Lie algebras*, Adv. Math. **285** (2015), 1066–1105.

[CNV10] S. Cecotti, A. Neitzke, and C. Vafa, *R-Twisting and 4d/2d Correspondences*, ArXiv e-prints (2010).

[FOS09] G. Fourier, M. Okado, and A. Schilling, *Kirillov-Reshetikhin crystals for nonexceptional types*, Adv. Math. **222** (2009), no. 3, 1080–1116. MR 2553378

[GOW16] M. J. Griffin, K. Ono, and S. O. Warnaar, *A framework of Rogers-Ramanujan identities and their arithmetic properties*, Duke Math. J. **165** (2016), no. 8, 1475–1527.

[Gra00] J. Gray, *Linear differential equations and group theory from Riemann to Poincaré*, second ed., Birkhäuser Boston, Inc., Boston, MA, 2000.

[GZ85] B. Gross and D. Zagier, *On singular moduli*, J. Reine Angew. Math. **355** (1985), 191–220.

[Her06] D. Hernandez, *The Kirillov-Reshetikhin conjecture and solutions of T-systems*, J. Reine Angew. Math. **596** (2006), 63–87.

[HKO+99] G. Hatayama, A. Kuniba, M. Okado, T. Takagi, and Y. Yamada, *Remarks on fermionic formula*, Recent developments in quantum affine algebras and related topics (Raleigh, NC, 1998), Contemp. Math., vol. 248, Amer. Math. Soc., Providence, RI, 1999, pp. 243–291.

[HL10] A. Huang and C.-h. Lee, *A note on Nahm's conjecture in rank 2 case*, Commun. Number Theory Phys. **4** (2010), no. 4, 609–622.

[Kel13] B. Keller, *The periodicity conjecture for pairs of Dynkin diagrams*, Ann. of Math. (2) **177** (2013), no. 1, 111–170. MR 2999039

[Kir89] A. N. Kirillov, *Identities for the Rogers dilogarithm function connected with simple Lie algebras*, Journal of Mathematical Sciences **47** (1989), 2450–2459.

[Kir95] ______, *Dilogarithm identities*, Progr. Theoret. Phys. Suppl. (1995), no. 118, 61–142, Quantum field theory, integrable models and beyond (Kyoto, 1994).

[KNS93] A. Kuniba, T. Nakanishi, and J. Suzuki, *Characters in conformal field theories from thermodynamic Bethe ansatz*, Modern Phys. Lett. A **8** (1993), no. 18, 1649–1659.

[KNS94] ______, *Functional relations in solvable lattice models. I. Functional relations and representation theory*, Internat. J. Modern Phys. A **9** (1994), no. 30, 5215–5266.

[KP84] V. G. Kac and D. H. Peterson, *Infinite-dimensional Lie algebras, theta functions and modular forms*, Adv. in Math. **53** (1984), no. 2, 125–264. MR 750341

[KR87] A. N. Kirillov and N. Yu. Reshetikhin, *Representations of Yangians and multiplicities of the inclusion of the irreducible components of the tensor product of representations of simple Lie algebras*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **160** (1987), no. Anal. Teor. Chisel i Teor. Funktsii. 8, 211–221, 301.

[Lee13a] C.-h. Lee, *Nahm's conjecture and Y-systems*, Commun. Number Theory Phys. **7** (2013), no. 1, 1–14.

[Lee13b] ______, *A proof of the KNS conjecture: D_r case*, J. Phys. A **46** (2013), no. 16, 165201, 12.

[Lee15] ______, *Linear recurrence relations in Q-systems and difference L-operators*, J. Phys. A **48** (2015), no. 19, 195201.

[Lee17] ______, *Positivity and periodicity of Q-systems in the WZW fusion ring*, Adv. Math. **311** (2017), 532 – 568.

[Lee18] ______, *Product formula for the limits of normalized characters of Kirillov-Reshetikhin modules*, arXiv e-prints (2018), arXiv:1807.11194.

[Lee19a] ______, *Linear recurrence relations in Q-systems via lattice points in polyhedra*, Transformation Groups (to appear) (2019).

[Lee19b] ______, *On polyhedral formula for Kirillov-Reshetikhin modules*, arXiv e-prints (2019), arXiv:1901.00104.

[Nah07] W. Nahm, *Conformal field theory and torsion elements of the Bloch group*, Frontiers in number theory, physics, and geometry. II, Springer, Berlin, 2007, pp. 67–132.

[Nak03] H. Nakajima, *t-analogs of q-characters of Kirillov-Reshetikhin modules of quantum affine algebras*, Represent. Theory **7** (2003), 259–274 (electronic).

[Nak11] T. Nakanishi, *Dilogarithm identities for conformal field theories and cluster algebras: simply laced case*, Nagoya Math. J. **202** (2011), 23–43.

[RW] E. M. Rains and S. O. Warnaar, *Bounded Littlewood identities*, Memoirs of the American Mathematical Society (to appear).

[VZ11] M. Vlasenko and S. Zwegers, *Nahm's conjecture: asymptotic computations and counterexamples*, Commun. Number Theory Phys. **5** (2011), no. 3, 617–642.

[Zag07] D. Zagier, *The dilogarithm function*, Frontiers in number theory, physics, and geometry. II, Springer, Berlin, 2007, pp. 3–65.

School of Mathematics, Korea Institute for Advanced Study, Seoul 130-722, Korea
E-mail address: chlee@kias.re.kr