Around Nahm's conjecture

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Introduction

Definition

Let A be a positive definite symmetric $r \times r$ matrix with rational entries, $B \in \mathbb{Q}^r$, $C \in \mathbb{Q}$. Consider an r-fold q-hypergeometric series given by

$$f_{A,B,C}(q) = \sum_{n=(n_1,\dots,n_r)\in(\mathbb{Z}_{\geq 0})^r} \frac{q^{\frac{1}{2}n^t An + B\cdot n + C}}{(q)_{n_1}\cdots(q)_{n_r}}$$

where $q \in \mathbb{C}$ with $|q| < 1$.

Question

When can $f_{A,B,C}$ be modular?

Nahm's equation

Let $A = (a_{ij})$. Consider the system of equations

$$x_i = \prod_{j=1}^{r} (1 - x_j)^{a_{ij}}$$

for $i = 1, \cdots, r$.

- there exists a unique real solution $\mathbf{x} = (x_i)$ such that $0 < x_i < 1$ for all $i = 1, \cdots r$.
- duality $(A, x) \mapsto (A^{-1}, 1 - x)$

Rogers dilogarithm function

The Rogers dilogarithm function is defined by

$$L(x) = \mathrm{Li}_2(x) + \frac{1}{2}\log x \log(1-x) = -\frac{1}{2}\int_0^x \frac{\log(1-y)}{y} + \frac{\log(y)}{1-y}dy$$

for $x \in (0,1)$. We set $L(0) = 0$ and $L(1) = \pi^2/6$ so that L is continuous on $[0,1]$.

asymptotic expansion

Let $\mathbf{x} = (x_i)$ be the solution of $\mathbf{x} = (1 - \mathbf{x})^A$ with $0 < x_i < 1$. As $t \searrow 0$,

$$f_{A,B,C}(e^{-t}) \sim \beta \exp(\frac{\alpha}{t} - \gamma t) \left(1 + \sum_{m=1}^{\infty} c_m t^m \right)$$

where

$$\alpha = \sum_i L(x_i)$$

and β, γ, c_m are certain numbers depending on A, B, C.

If $f_{A,B,C}$ is a modular form of weight k,

- $k = 0$
- $\alpha = \sum_i L(x_i)$ must be a rational multiple of $L(1) = \pi^2/6$
- $\cdots$

Rogers-Ramanujan identities

$$G(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}$$

$$H(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}.$$

$q^{-1/60}G(q)$ and $q^{11/60}H(q)$ are modular. In this case, we have modular triples

$$((2),(0),-1/60)$$

and

$$((2),(1),11/60)$$

Rogers-Ramanujan identities

In the case of the Rogers-Ramanujan identities, the asymptotics gives

$$G(q), H(q) \sim \exp\left(\frac{\pi^2}{15t}\right).$$

Note that the number $\frac{3-\sqrt{5}}{2}$ is a solution of $x = (1-x)^2$. The number $\frac{\pi^2}{15}$ inside the exponential term is obtained from the identity

$$L\left(\frac{3-\sqrt{5}}{2}\right) = \frac{\pi^2}{15} = \frac{2}{5}L(1).$$

Andrews-Gordon identities

For two integers k, i such that $k \geq 2$ and $1 \leq i \leq k$,

$$\sum_{n_1,\cdots,n_{k-1}\geq 0} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_i+\cdots+N_{k-1}}}{(q)_{n_1}\dots(q)_{n_{k-1}}} = \prod_{r\neq 0,\pm i \pmod{2k+1}} \frac{1}{1-q^r}$$

where if $1 \leq j \leq k-1$, $N_j = n_j + \cdots + n_{k-1}$ and if $j = k$, $N_j = 0$.
$k = 2$ recovers the Rogers-Ramanujan identities

Andrews-Gordon identities

When $k = 3$ and $i = 1$,

$$\sum_{n_1, n_2 \geq 0} \frac{q^{n_1^2+2n_1 n_2+2n_2^2+n_1+2n_2}}{(q)_{n_1}(q)_{n_2}} = \frac{(q;q^7)_\infty (q^6;q^7)_\infty (q^7;q^7)_\infty}{(q)_\infty}.$$

a modular triple

$$(A, (1,2), 17/42)$$

where

$$A = \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}.$$

The solution of $x = (1-x)^A$ with $0 < x_i < 1$ gives

$$L(x_1) + L(x_2) = \frac{4}{7} L(1).$$

Bloch group

Let F be a field. $\Lambda^2 F^*$ denotes the abelian group of formal sums of $x \wedge y$ for $x, y \in F^*$ modulo the relations

$$\begin{aligned} x \wedge x &= 0 \\ (x_1 x_2) \wedge y &= x_1 \wedge y + x_2 \wedge y \\ x \wedge (y_1 y_2) &= x \wedge y_1 + x \wedge y_2 \end{aligned}$$

Let $\partial : \mathbb{Z}[F^* \backslash \{1\}] \to \Lambda^2(F^*)$ be a $\mathbb{Z}$-linear map defined by

$$\partial([x]) = x \wedge (1 - x).$$

Bloch group

Let $A(F) = \ker \partial$ and $C(F)$ the subgroup of $A(F)$ generated by the elements

$$[x] + [1 - xy] + [y] + [\frac{1-y}{1-xy}] + [\frac{1-x}{1-xy}]$$
$$[x] + [1 - x]$$
$$[x] + [\frac{1}{x}]$$

Definition

The Bloch group $\mathcal{B}(F)$ of F is defined by $\mathcal{B}(F) = A(F)/C(F)$.

Nahm's conjecture : statement

For $x = (x_1, ..., x_r) \in F^r$, let $\xi_x = \sum_{i=1}^{r} [x_i] \in \mathbb{Z}[F]$

Conjecture

Let A be a positive definite symmetric $r \times r$ matrix with rational entries. The followings are equivalent:

(i) *For any solution $x = (x_1, ..., x_r) \in F^r$ of $\mathbf{x} = (1 - \mathbf{x})^A$, the element $\xi_x \in \mathcal{B}(F)$ is a torsion element of $\mathcal{B}(F)$.*

(ii) *There exists a modular triple (A, B, C).*

Nahm's conjecture : status

- rank 1 case : complete classification (Zagier)

A	B	C
2	0	-1/60
2	1	11/60
1	0	-1/48
1	1/2	1/24
1	-1/2	1/24
1/2	0	-1/40
1/2	1/2	1/40

- counterexamples found (Vlasenko and Zwegers)
- open problem : fix the conjecture !

Kronecker product of two Cartan matrices of type *ADET*

Let us consider matrices of the form $A = \mathcal{C}(X) \otimes \mathcal{C}(X')^{-1}$ where $\mathcal{C}(X), \mathcal{C}(X')$ are Cartan matrices of type A, D, E, T. Why?

- rank 1 case
- integral matrices in rank 2 case
- Andrews-Gordon identities has matrix $A = \mathcal{C}(A_1) \otimes \mathcal{C}(T_n)^{-1}$

$$\mathcal{C}(A_1) \otimes \mathcal{C}(T_n)^{-1} = \begin{pmatrix} 2 & 2 & 2 & 2 & 2 \\ 2 & 4 & 4 & 4 & 4 \\ 2 & 4 & 6 & 6 & 6 \\ 2 & 4 & 6 & \ddots & \vdots \\ 2 & 4 & 6 & \cdots & 2n \end{pmatrix}.$$

- connections with other mathematics like cluster algebras

the dilogarithm function

Definition

The dilogarithm function is defined by

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t} dt$$

for $z \in \mathbb{C} - [1, \infty)$.

For $|z| < 1$, we have the following power series for the dilogarithm function :

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$

Bloch-Wigner dilogarithm

Definition

The Bloch-Wigner dilogarithm function is given by

$$D(z) = \mathrm{Im}(\mathrm{Li}_2(z)) + \log|z| \arg(1-z).$$

- $D(z)$ is a real analytic function on $\mathbb{C}$ except at 0 and 1, where it is continuous but not differentiable.
- $D(\bar{z}) = -D(z)$, it vanishes on $\mathbb{R}$.
- functional equations :

$$D(x) + D(1-xy) + D(y) + D(\tfrac{1-y}{1-xy}) + D(\tfrac{1-x}{1-xy}) = 0$$
$$D(x) + D(1-x) = 0$$
$$D(x) + D(\tfrac{1}{x}) = 0$$

functional dilogarithm identities

Suppose that we have a set S of rational functions $f_i \in \mathbb{C}(y_1, \cdots, y_n)$ such that

$$\sum_{f_i \in S} f_i \wedge (1 - f_i) = 0 \in \Lambda^2 \mathbb{C}(y_1, \cdots, y_n)^*.$$

Then $\sum_{f_i \in S} D(f_i|_{\mathbf{x}})$ is independent of $\mathbf{x} \in \mathbb{C}^n$.

regulator map

The Bloch-Wigner dilogarithm $D(z)$ can be used to define a map

$$D : \mathcal{B}(\mathbb{C}) \to \mathbb{R}$$

$$\xi = \sum_i n_i[x_i] \mapsto D(\xi) = \sum_i n_i D(x_i)$$

Let F be a number field of degree $r_1 + 2r_2$ over $\mathbb{Q}$ (r_1 real and $2r_2$ complex embeddings). Then

$$\mathcal{B}(F)/(\text{torsion}) \to \mathbb{R}^{r_2}$$

$$\xi \mapsto (D(\sigma_1(\xi)), \cdots, D(\sigma_{r_2}(\xi)))$$

is an isomorphism onto a lattice of finite covolume.

Y-system:definition

Definition

For a family of variables, $\{Y_{ii'}(u) | i \in I, i' \in I', u \in \mathbb{Z}\}$, the Y-system $\mathbb{Y}(X, X')$ associated with a pair (X, X') of ADE Dynkin diagrams is defined as a system of recurrence relations as follows :

$$Y_{ii'}(u-1)Y_{ii'}(u+1) = \frac{\prod_{j \in I}(1 + Y_{ji'}(u))^{\mathcal{I}(X)_{ij}}}{\prod_{j' \in I'}\left(1 + Y_{ij'}(u)^{-1}\right)^{\mathcal{I}(X')_{i'j'}}}$$

Y-system : $\mathbb{Y}(A_2, A_1)$ example

$$Y_i(u-1)Y_i(u+1) = \prod_{j:j\sim i}(1+Y_j(u)).$$

u	$Y_1(u)$	$Y_2(u)$
-1	$\frac{1}{\alpha}$	x
0	y	β
1	$\alpha(\beta+1)$	$\frac{y+1}{x}$
2	$\frac{x+y+1}{xy}$	$\alpha+\frac{\alpha+1}{\beta}$
3	$\frac{\alpha+1}{\alpha\beta}$	$\frac{x+1}{y}$
4	x	$\frac{1}{\alpha}$
5	β	y
6	$\frac{y+1}{x}$	$\alpha(\beta+1)$
$\vdots$	$\vdots$	$\vdots$

Y-system : $\mathbb{Y}(A_2, A_1)$ example

$$S = \left\{ x, y, \frac{y+1}{x}, \frac{x+y+1}{xy}, \frac{x+1}{y} \right\}$$

- $r = 2, h = 3$ and $r' = 1, h' = 2$.
- all of them are Laurent polynomials in x and y.
- functional dilogarithm identities

$$\begin{aligned}
\sum_{a \in S} L\left(\frac{a}{1+a}\right) \\
= L\left(\frac{x}{x+1}\right) + L\left(\frac{y}{y+1}\right) + L\left(\frac{y+1}{x+y+1}\right) \\
+ L\left(\frac{x+y+1}{xy+x+y+1}\right) + L\left(\frac{x+1}{x+y+1}\right) \\
= 3L(1) = \frac{\pi^2}{2}
\end{aligned}$$

Y-system : properties

- periodicity (Keller) : $Y_{\mathbf{i}}(u+2(h+h')) = Y_{\mathbf{i}}(u)$
- Laurent property
- constancy condition (Nakanishi)

$$\sum_{(\mathbf{i},u)\in S_+} Y_{\mathbf{i}}(u) \wedge (1+Y_{\mathbf{i}}(u)) = 0 \in \Lambda^2 \mathbb{Q}(y)^*.$$

- functional dilogarithm identity

$$\sum_{(\mathbf{i},u)\in S_+} L\left(\frac{Y_{\mathbf{i}}(u)}{1+Y_{\mathbf{i}}(u)}\right) = hr r' L(1)$$

or,

$$\sum_{(\mathbf{i},u)\in S_+} D\left(\frac{Y_{\mathbf{i}}(u)}{1+Y_{\mathbf{i}}(u)}\right) = 0.$$

ADET Cartan matrices and torsion in the Bloch group

Theorem

Let $A = \mathcal{C}(X) \otimes \mathcal{C}(X')^{-1}$ where (X, X') is a pair of Dynkin diagrams of type ADET. For every solution $\mathbf{x} = (x_\mathbf{i})_{\mathbf{i} \in \mathbf{I}}$ of the equation $\mathbf{x} = (1 - \mathbf{x})^A$ in a number field F, $\xi_\mathbf{x} = \sum_{\mathbf{i} \in \mathbf{I}} [x_\mathbf{i}]$ is a torsion element of the Bloch group $\mathcal{B}(F)$.

sketch of the proof

The system of equations $\mathbf{x} = (1-\mathbf{x})^A$ can be rewritten as

$$\prod_{j' \in I'} x_{ij'}^{\mathcal{C}(X')_{i'j'}} = \prod_{j \in I} (1 - x_{ji'})^{\mathcal{C}(X)_{ij}}, (i, i') \in \mathbf{I}.$$

Use the change of variables $y_{\mathbf{i}} = \frac{x_{\mathbf{i}}}{1-x_{\mathbf{i}}}$ or $x_{\mathbf{i}} = \frac{y_{\mathbf{i}}}{1+y_{\mathbf{i}}} = \frac{1}{1+y_{\mathbf{i}}^{-1}}$.

$$\prod_{j' \in I'} \left(\frac{1}{1+y_{ij'}^{-1}}\right)^{\mathcal{C}(X')_{i'j'}} = \prod_{j \in I} \left(\frac{1}{1+y_{ji'}}\right)^{\mathcal{C}(X)_{ij}}$$

$$1 = \frac{\prod_{j \in I} (1+y_{ji'})^{-\mathcal{C}(X)_{ij}}}{\prod_{j' \in I'} (1+y_{ij'}^{-1})^{-\mathcal{C}(X')_{i'j'}}}.$$

sketch of the proof

This can be written as

$$\left(\frac{1}{1+y_{ii'}^{-1}}\right)^2 (1+y_{ii'})^2 = \frac{\prod_{j\in I}(1+y_{ji'})^{\mathcal{I}(X)_{ij}}}{\prod_{j'\in I'}\left(1+y_{ij'}^{-1}\right)^{\mathcal{I}(X')_{i'j'}}}.$$

This is just the constant Y-system

$$y_{ii'}^2 = \frac{\prod_{j\in I}(1+y_{ji'})^{\mathcal{I}(X)_{ij}}}{\prod_{j'\in I'}\left(1+y_{ij'}^{-1}\right)^{\mathcal{I}(X')_{i'j'}}}.$$

This implies

$$\sum_{\mathbf{i}\in\mathbf{I}} D\left(\sigma(x_{\mathbf{i}})\right) = 0$$

for any $\sigma : F \hookrightarrow \mathbb{C}$.

from Nahm's equation to Q-system

By applying the change of variables

$$x_{ii'} = \prod_{j \in I} z_{ji'}^{-\mathcal{C}(X)_{ij}}$$

to

$$\prod_{j' \in I'} x_{ij'}^{\mathcal{C}(X')_{i'j'}} = \prod_{j \in I} (1 - x_{ji'})^{\mathcal{C}(X)_{ij}}, (i, i') \in \mathbf{I}$$

we get another system of equations

$$z_{ii'}^2 = \prod_{j \in I} z_{ji'}^{\mathcal{I}(X)_{ij}} + \prod_{j' \in I'} z_{ij'}^{\mathcal{I}(X')_{i'j'}}, (i, i') \in \mathbf{I}.$$

Q-system : definition

Definition

Let X be a Dynkin diagram of type ADE. For a family of variables $\{Q_i^{(a)} | a \in I, i \in \mathbb{Z}^{\geq 0}\}$, consider recurrences given by

$$\left(Q_i^{(a)}\right)^2 = \prod_{b \in I} \left(Q_i^{(b)}\right)^{\mathcal{I}(X)_{ab}} + Q_{i-1}^{(a)} Q_{i+1}^{(a)} \tag{4.1}$$

We call this system of recurrence equations the unrestricted Q-system of type X. We use boundary conditions $Q_0^{(a)} = 1$ for all $a \in I$.

level k restricted Q-system

Definition

We call the following system of equations

$$\begin{cases} Q_0^{(a)} = 1 & a \in I \\ \left(Q_i^{(a)}\right)^2 = \prod_{b\in I} \left(Q_i^{(b)}\right)^{\mathcal{I}(X)_{ab}} + Q_{i-1}^{(a)} Q_{i+1}^{(a)} & 1 \le i < k, a \in I \\ Q_k^{(a)} = 1 & a \in I \end{cases} \tag{4.2}$$

for variables $\left(Q_i^{(a)}\right)$ with $0 \le i \le k$ and $a \in I$ the level k restricted Q-system of type X.

(X, k) corresponds to (X, A_{k-1}) pair of Dynkin diagrams.

character solution of the unrestricted Q-systems

For $X = A_r$, we have $Q_i^{(a)} = \chi_{i\omega_a}$ for $a \in I$ and $i \in \mathbb{Z}^{\geq 0}$ and they satisfy the unrestricted Q-system of type A_r.
For $X = D_r$,

$$Q_i^{(a)} = \begin{cases} \sum \chi_{k_a\omega_a + k_{a-2}\omega_{a-2} + \cdots + k_1\omega_1} & 1 \leq a < r-1, a \equiv 1 \pmod 2, \\ \sum \chi_{k_a\omega_a + k_{a-2}\omega_{a-2} + \cdots + k_0\omega_0} & 1 \leq a < r-1, a \equiv 0 \pmod 2, \\ \chi_{m\omega_a} & a = r-1, r \end{cases}$$

where $\omega_0 = 0$ and the summation is over all nonnegative integers satisfying $k_a + k_{a-2} + \cdots + k_1 = i$ for a odd and $k_a + k_{a-2} + \cdots + k_0 = i$ for a even.

Kuniba-Nakanishi-Suzuki (KNS) conjecture

Conjecture

Let $z_i^{(a)} = Q_i^{(a)}(\frac{\rho}{h+k})$ for $a \in I$ and $i \in \mathbb{Z}^{\geq 0}$. It satisfies the following properties :

- *(positivity) $z_i^{(a)} > 0$ for $0 \leq i \leq k$ and $a \in I$.*
- *(symmetry) $z_i^{(a)} = z_{k-i}^{(a)}$ for $1 \leq i \leq k-1$ and $a \in I$.*
- *(unit boundary condition) $z_k^{(a)} = 1$ for $a \in I$.*
- *(unimodality) $z_{i-1}^{(a)} < z_i^{(a)}$ holds true for $1 \leq i \leq m = \lfloor \frac{k}{2} \rfloor$ and $a \in I$ where $\lfloor x \rfloor$ is the floor function.*
- *(occurrence of 0) $z_{k+1}^{(a)} = z_{k+2}^{(a)} = \cdots = z_{k+h-1}^{(a)} = 0$ for $a \in I$.*

It has been known to be true for $X = A_r$.

KNS conjecture : $X = D_r$ case

Theorem

The KNS conjecture is true for $X = D_r$. Moreover, $z_{k+h}^{(a)} = 1$ for $1 \leq a \leq r-2$ and $z_{k+h}^{(r-1)} = z_{k+h}^{(r)} = \pm 1$.

key ingredients : the affine Weyl group symmetry of quantum dimensions of affine weights
implication to Nahm's conjecture : solutions of Nahm's equation landed in the modular tensor category!

example : $X = D_5, k = 4$

For $a = 1, 4, 5$ and $1 \le i \le h + k = 12$,

$$\begin{bmatrix} z_1^{(1)} & z_1^{(4)} & z_1^{(5)} \\ z_2^{(1)} & z_2^{(4)} & z_2^{(5)} \\ z_3^{(1)} & z_3^{(4)} & z_3^{(5)} \\ z_4^{(1)} & z_4^{(4)} & z_4^{(5)} \\ z_5^{(1)} & z_5^{(4)} & z_5^{(5)} \\ z_6^{(1)} & z_6^{(4)} & z_6^{(5)} \\ z_7^{(1)} & z_7^{(4)} & z_7^{(5)} \\ z_8^{(1)} & z_8^{(4)} & z_8^{(5)} \\ z_9^{(1)} & z_9^{(4)} & z_9^{(5)} \\ z_{10}^{(1)} & z_{10}^{(4)} & z_{10}^{(5)} \\ z_{11}^{(1)} & z_{11}^{(4)} & z_{11}^{(5)} \\ z_{12}^{(1)} & z_{12}^{(4)} & z_{12}^{(5)} \end{bmatrix} = \begin{bmatrix} \mathcal{D}_{3\hat\omega_0+\hat\omega_1} & \mathcal{D}_{3\hat\omega_0+\hat\omega_4} & \mathcal{D}_{3\hat\omega_0+\hat\omega_5} \\ \mathcal{D}_{2\hat\omega_0+2\hat\omega_1} & \mathcal{D}_{2\hat\omega_0+2\hat\omega_4} & \mathcal{D}_{2\hat\omega_0+2\hat\omega_5} \\ \mathcal{D}_{\hat\omega_0+3\hat\omega_1} & \mathcal{D}_{\hat\omega_0+3\hat\omega_4} & \mathcal{D}_{\hat\omega_0+3\hat\omega_5} \\ \mathcal{D}_{4\hat\omega_1} & \mathcal{D}_{4\hat\omega_4} & \mathcal{D}_{4\hat\omega_5} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathcal{D}_{4\hat\omega_1} & \mathcal{D}_{4\hat\omega_4} & \mathcal{D}_{4\hat\omega_5} \end{bmatrix}.$$

example : $X = D_5, k = 4$

For $a = 2, 3$ and $1 \leq i \leq h + k = 12$,

$$\begin{bmatrix} z_1^{(2)} & z_1^{(3)} \\ z_2^{(2)} & z_2^{(3)} \\ z_3^{(2)} & z_3^{(3)} \\ z_4^{(2)} & z_4^{(3)} \\ z_5^{(2)} & z_5^{(3)} \\ z_6^{(2)} & z_6^{(3)} \\ z_7^{(2)} & z_7^{(3)} \\ z_8^{(2)} & z_8^{(3)} \\ z_9^{(2)} & z_9^{(3)} \\ z_{10}^{(2)} & z_{10}^{(3)} \\ z_{11}^{(2)} & z_{11}^{(3)} \\ z_{12}^{(2)} & z_{12}^{(3)} \end{bmatrix} = \begin{bmatrix} \mathcal{D}_{4\hat{\omega}_0} + \mathcal{D}_{2\hat{\omega}_0+\hat{\omega}_2} & \mathcal{D}_{3\hat{\omega}_0+\hat{\omega}_1} + \mathcal{D}_{2\hat{\omega}_0+\hat{\omega}_3} \\ \mathcal{D}_{4\hat{\omega}_0} + \mathcal{D}_{2\hat{\omega}_2} + \mathcal{D}_{2\hat{\omega}_0+\hat{\omega}_2} & \mathcal{D}_{2\hat{\omega}_0+2\hat{\omega}_1} + \mathcal{D}_{2\hat{\omega}_3} + \mathcal{D}_{\hat{\omega}_0+\hat{\omega}_1+\hat{\omega}_3} \\ \mathcal{D}_{4\hat{\omega}_0} + \mathcal{D}_{2\hat{\omega}_0+\hat{\omega}_2} & \mathcal{D}_{\hat{\omega}_0+3\hat{\omega}_1} + \mathcal{D}_{2\hat{\omega}_1+\hat{\omega}_3} \\ \mathcal{D}_{4\hat{\omega}_0} & \mathcal{D}_{4\hat{\omega}_1} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \mathcal{D}_{4\hat{\omega}_0} & \mathcal{D}_{4\hat{\omega}_1} \end{bmatrix}.$$

future directions?

- cluster algerbras
- wall-crossing invariants
- quantum dilogarithms
- categorical interpretation of Nahm's equation

Thank you!