Research Statement

Chul-hee Lee

My research interests are at the interface of representation theory, number theory and mathematical physics. In particular, there are two very different categories that my research has been centered around : the category of finite-dimensional representations of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ which is an example of quantum groups, and the category of certain infinite-dimensional representations of the affine Lie algebra $\widehat{\mathfrak{g}}$ which is an infinite-dimensional Lie algebra. In Section 1, I will discuss some results [Lee19a, Lee19b, Lee19c] on important representations of $U_q(\widehat{\mathfrak{g}})$. In Section 2, I will explain a connection between these two categories discovered in [Lee17]. In the last section, I will talk about my current and future research.

General overview of quantum affine algebras

Quantum affine algebras are an important class of quantum groups with rich representations. The theory of quantum groups has emerged from the study of integrable systems and exactly solvable models in mathematical physics. Various branches of theoretical physics contributed to this development : Bethe's method (Bethe Ansatz) for quantum mechanical models of magnets, Baxter's work on statistical models, and inverse scattering method for certain differential equations admitting soliton solutions. The search for common algebraic structures behind these approaches eventually led to the discovery of quantum groups in the 80s by Drinfeld and Jimbo. Since then, their representations have become an important area of research. In essence, quantum groups are certain algebraic structures that are built to generate the solutions of the Yang-Baxter equation. Since its inception, the theory has also been closely tied with the invariants of knots and links, such as the Jones polynomials via the Yang-Baxter equation.

General overview of affine Lie algebras

Affine Lie algebras are certain infinite-dimensional Lie algebras, being particular examples of Kac-Moody algebras. They play an important role in string theory and two-dimensional conformal field theory. One of striking and mysterious phenomena related to affine Lie algebras is that the characters of their integrable representations transform amongst themselves under the modular group. Monstrous moonshine, an unexpected connection between the monster group and modular functions, although requiring a different setup, shares many ingredients with the theory of affine Lie algebras. This kind of modularity arising from Lie theory has been one of my long-standing interests in mathematics.

1 Kirillov-Reshetikhin modules

Context

The Kirillov-Reshetikhin (KR) modules form a special class of finite-dimensional representations of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$. They were introduced by Kirillov and Reshetikhin in the pioneering work [KR87], bringing Bethe Ansatz from mathematical physics to the representation theory of quantum affine algebras and Yangians.

They formulated a conjectural rule to decompose a tensor product of KR modules when it is restricted to a subalgebra of $U_q(\widehat{\mathfrak{g}})$. Representation theoretically speaking, they conjectured a branching rule guided by Bethe's original work. This was the beginning of the KR conjecture, and it soon became one of the central problems in the representation theory of quantum affine algebras. The attempt to find its rigorous proof has led to a series of developments of important tools and concepts in the theory.

Later, Kuniba and his collaborators came up with a strategy for [HKO+99, KNT02] to prove the KR conjecture. Roughly speaking, it consists of proving the following two statements :

1. the characters of KR modules satisfy a non-linear difference equation called the Q-system
2. normalized characters of KR modules have a limit as a formal power series as $m \to \infty$ for fixed $a \in I$ (KR modules are labeled by a pair $(a, m) \in I \times \mathbb{Z}_{\geq 0}$ for some finite set I)

This strategy has been successfully carried out by Nakajima [Nak03] (ADE-types) and Hernandez [Her06]. One of the key elements in their proof is the T-system of Kuniba-Nakanishi-Suzuki [KNS94], which is another non-linear difference equation generalizing the Q-system. The original KR conjecture has been completely settled by now.

KR modules have recently gained more importance beyond representation theory. A notable one is the recent program of 'monoidal categorification of cluster algebras' proposed by Hernandez and Leclerc [HL10]. Here the T-system provides an important link. This has led to many profound discoveries such as the recent works of Kang, Kashiwara, Kim and Oh [KKK18, KKKO18]. I am also very much interested in this new development.

My results

The above-mentioned convergence of normalized characters of KR modules can be considered as a fundamental result about KR modules. An explicit formula for it has started getting attention relatively recently, mainly after the work [HJ12] by Hernandez and Jimbo. They found a way to regard this limit of characters as a character of a certain limit module which turns out be very closely related to Baxter's classic works, as clarified by Frenkel and Hernandez [FH15]. A very compact and explicit formula soon appeared as a conjecture of Mukhin and Young [MY14]. As characters contain lots of structural information about representations, it is regarded as an important problem to find an explicit formula for the character of a given representation.

I recently obtained a proof of the above conjectural formula of Mukhin and Young, except for a small number of cases [Lee19c]. The formula is nicely given as a certain product over the positive roots of $\mathfrak{g}$, resembling Weyl's famous denominator formula. I found new algebraic relations satisfied by limits of characters, and these relations are sufficiently strong to identify them with conjectural forms.

The proof was partly based on the results from [Lee19a], which was the study of structural properties of certain linear recurrences in the sequence of characters of KR modules. It turns out to be a quite useful tool in investigating some fundamental properties of KR modules. This was originally developed to understand certain problems posed in [Lee17] which is explained in the next section in more detail. Recently, I found another application of it to the branching problem of KR modules [Lee19b]. I believe that there are more applications of it to KR modules. Some cases of those linear recurrences have not been settled, and it is necessary to complete them to expand its applicability.

2 Quantum affine algebras and fusion rings (Verlinde rings)

Context

A conjecture formulated by Kirillov [Kir89] and Kuniba-Nakanishi-Suzuki [KNS94] claims that one can solve a certain system of equations appearing in mathematical physics by appropriately specializing the characters of KR modules. Let us call it the **KKNS conjecture** here. This system consists of a finite set of algebraic equations in the same number of variables over the complex numbers, and has a unique positive real solution. These positive numbers originally showed up to express certain important physical quantities in terms of the dilogarithm function in the context of thermodynamic Bethe Ansatz.

The key part of the conjecture was that this unique positive solution of the system is given by the so called quantum dimensions of KR modules. As quantum dimensions are algebraic numbers which are not necessarily positive, proving their positivity is not trivial. The conjecture had been open except for some very simple cases without any essential progress.

It might be useful to add a few more general remarks on mathematical aspects of thermodynamic Bethe Ansatz. It refers to a set of methods used in mathematical physics to compute certain thermodynamic quantities of physical models based on Bethe Ansatz. Although its precise mathematical formulation still seems to be missing, it often produces mathematically provable statements. And they are considered important because their proof usually require new mathematical ideas and objects. A notable example is Keller's recent proof [Kel13] of the periodicity of certain difference equations conjectured by Zamolodchikov [Zam91]. These new objects and ideas discovered during the quest for proofs will ultimately help us understand more precise mathematical structures underlying the thermodynamic Bethe Ansatz.

My results

In [Lee17], I managed to prove the **KKNS conjecture** for all classical types. The most important contribution of this work has been to introduce the fusion ring, also known as the Verlinde ring, into the problem. The fusion ring arises from the category of integrable representations of the affine Lie algebra $\widehat{\mathfrak{g}}$ of given level equipped with the fusion product. It is an important object in many areas of mathematics and physics, including algebraic geometry, conformal field theory and topological quantum field theory [Bea96, BK01, Koh02]. The fusion product of two such representations is one of deep concepts originating from conformal field theory, and is very different from the tensor product as representations of a Lie algebra.

Let me briefly explain the key construction which involves two ring homomorphisms. The first is given by the homomorphism $\mathcal{R}(U_q(\widehat{\mathfrak{g}})) \xrightarrow{\text{res}} \mathcal{R}(U_q(\mathfrak{g}))$ coming from restriction. Here $\mathcal{R}(\cdot)$ denotes the Grothendieck ring of the category of finite-dimensional representations of relevant quantum groups. The fusion ring $\mathcal{R}(\widehat{\mathfrak{g}}, k)$ depends on the choice of a positive integer k called the level. As a ring, it is a free $\mathbb{Z}$-module of finite rank with a distinguished basis. There is a surjective homomorphism $\mathcal{R}(U_q(\mathfrak{g})) \xrightarrow{\beta_k} \mathcal{R}(\widehat{\mathfrak{g}}, k)$ playing an important role in its study [Tel95]. According to Teleman, its existence was originally conjectured by Bott from structures of Verlinde's formula. In physics literature, it is often called the Kac-Walton formula.

It turns out that it is essential to study the composition $\mathcal{R}(U_q(\widehat{\mathfrak{g}})) \xrightarrow{\text{res}} \mathcal{R}(U_q(\mathfrak{g})) \xrightarrow{\beta_k} \mathcal{R}(\widehat{\mathfrak{g}}, k)$ of these two homomorphisms, which had been studied separately and not been considered in this way before my work. I was able to reformulate the KKNS conjecture in terms of the fusion ring, and it makes many aspects of the problem clear. Especially, numbers are replaced by certain objects in a category (or corresponding elements in a ring) so that the positivity question can be considered in an upgraded setting with more tools. This formulation and its partial progress led to a proof of the original conjecture and a set of new conjectures.

Some results explained in the previous section have emerged from the issues presented in this work, but there are still interesting unresolved problems. The positivity problem in the fusion ring setting is a completely new and open problem proposed in [Lee17]. Although I expect that there is an explicit formula exhibiting this positivity in a combinatorial way, its exact form is elusive yet. This seems to be a fundamental problem in representation theory and mathematical physics, and more work is required.

3 Current and future research

3.1 Branching identities for Macdonald-Koornwinder polynomials

In [BW15, GOW16, RW], Warnaar and his collaborators obtained a new family of Rogers-Ramanujan identities. A crucial new ingredient found in [RW] by Rains and Warnaar is that suitable branching identities for Macdonald polynomials can clarify and simplify important steps. Macdonald polynomials are an important family of symmetric polynomials. They are central objects in algebraic combinatorics, and also have deep relationships with affine Hecke algebras, and with Hilbert schemes in algebraic geometry in special cases. In a collaboration with Warnaar, I have found new conjectural branching identities for Macdonald polynomials along this line. This is a work in progress.

3.2 (q,t) Littlewood-Richardson coefficients

Another interesting and fundamentally important problem in symmetric polynomials is finding the coefficients arising from the expansion of product of two symmetric polynomials with respect to a chosen basis. For example, the coefficients for product of two Schur functions in the Schur basis are called the Littlewood-Richardson coefficients. From the representation theoretic point of view, these numbers are the multiplicities in the decomposition of the tensor product of two irreducible representations of $\mathrm{GL}_n(\mathbb{C})$ into irreducibles.

When this problem is considered for Macdonald polynomials, the resulting coefficients are not necessarily numbers but rational functions in $\mathbb{Q}(q,t)$. With Warnaar, I observed that these rational functions sometimes exhibit very interesting and structural patterns, especially when the corresponding original Littlewood-Richardson coefficients are 1. This seems to be quite significant, but there is not much known about it. There is an ongoing collaboration with Warnaar on this issue.

3.3 Symmetric polynomials and p-adic groups

The study of symmetric polynomials such as Macdonald polynomials is now often considered as an independent research area of algebraic combinatorics. Unfortunately, the classical connections with the p-adic theory, as found in Macdonald's early work, are easily forgotten. In the early 70s, Macdonald proved a formula for the spherical functions on p-adic groups in terms of Hall-Littlewood symmetric polynomials. This formula was later revisited and expanded by Casselman and Shalika from the viewpoint of representations of p-adic groups.

In a collaboration with Warnaar, some novel conjectural vanishing phenomena for symmetric polynomials have been observed. There is still no good explanation for it, and I believe this is a phenomenon originating from spherical functions on p-adic symmetric spaces. It should be an interesting future project.

3.4 computational approach to Siegel modular forms

The Gross-Keating invariant of a half-integral matrix over a p-adic integer ring is a fundamental concept in the study of quadratic forms. In [IK16], Ikeda and Katsurada showed that the Gross-Keating invariant of B determines a certain polynomial invariant of B that appears as a local factor in the Fourier coefficients of Siegel-Eisenstein series. Through Garrett's pullback formula, these coefficients can be used for computing other Siegel modular forms, and eventually, the automorphic L-functions attached to cusp forms. It demonstrates why the Gross-Keating invariant is important in a broader context.

Based on a work by Cho, Ikeda, Katsurada, and Yamauchi, I presented the complete details of algorithms to compute the Gross-Keating invariants and some related quantities in [Lee18]. I also implemented them as a publicly available Mathematica package. This is the first computer implementation for Gross-Keating invariants. It can serve as a basis for a more comprehensive software for quadratic forms and Siegel modular forms. I expect it might help to investigate some open problems in Siegel modular forms such as the Miyawaki lift in the future.

3.5 Modular q-hypergeometric series in representation theory

Finding a criterion for a q-hypergeometric series to be a modular function is a widely-open problem in number theory. Nahm's conjecture addresses this question [Nah07] whose inspiration is found in thermodynamic Bethe Ansatz. I have been searching for functions arising from Lie theory, which fit into Nahm's framework after my work [Lee13]. Some results described in the previous sections actually arose along this line of pursuits. The conjectural candidate functions are the string functions of affine Lie algebras. There are still many unresolved problems around them such as the conjecture of Kuniba-Nakanishi-Suzuki, and it will be a fruitful area of further research in representation theory involving two categories I mentioned in the beginning.

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