AN EXPLICIT FORMULA FOR SOME VECTOR-VALUED DIFFERENTIAL OPERATOR ON SIEGEL MODULAR FORMS AND ITS APPLICATIONS

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Abstract. Let $\mathbf{k} = (k, \overbrace{l, \dots, l}^{n-1})$ with $k \geq l$. In this paper, we give an explicit formula for the differential operator which sends $M_{l-2}(\mathrm{Sp}_{2n}(\mathbb{Z}))$ to $S_{\mathbf{k}}(\mathrm{Sp}_n(\mathbb{Z})) \otimes S_{\mathbf{k}}(\mathrm{Sp}_n(\mathbb{Z}))$. As an application, we give an algorithm for computing the special value of the standard L-function at $s = l - n - 2$ of a Hecke eigenform in $S_{\mathbf{k}}(\mathrm{Sp}_n(\mathbb{Z}))$. Moreover, we apply our differential operator to produce a congruence between vector-valued cuspidal Hecke eigenforms.

1. Introduction

For a non-increasing sequence $\mathbf{k} = (k_1, \dots, k_n)$ of non-negative integers, let $M_{\mathbf{k}}(\mathrm{Sp}_n(\mathbb{Z}))$ be the space of modular forms of weight $\mathbf{k}$ for $\mathrm{Sp}_n(\mathbb{Z})$ and $S_{\mathbf{k}}(\mathrm{Sp}_n(\mathbb{Z}))$ its subspace of cusp forms. In particular, if $\mathbf{k} = (k, \dots, k)$, we simply write $M_k(\mathrm{Sp}_n(\mathbb{Z})) = M_{\mathbf{k}}(\mathrm{Sp}_n(\mathbb{Z}))$ and $S_k(\mathrm{Sp}_n(\mathbb{Z})) = S_{\mathbf{k}}(\mathrm{Sp}_n(\mathbb{Z}))$. Let $\boldsymbol{\ell} = (l_1, l_2, \dots)$ be a finite or infinite sequence of non-negative integers of depth m. (For the definition of depth, see Section 2.) For integers $n_1, n_2 \geq m$ and a positive integer k_0, put $\mathbf{k}_1 = (l_1 + k_0, \dots, l_{n_1} + k_0), \mathbf{k}_2 = (l_1 + k_0, \dots, l_{n_2} + k_0)$, and let $\widetilde{\mathbb{D}}_{k_0, \boldsymbol{\ell}, n_1, n_2}$ be the differential operator

$$M_{k_0}(\mathrm{Sp}_{n_1+n_2}(\mathbb{Z})) \to M_{\mathbf{k}_1}(\mathrm{Sp}_{n_1}(\mathbb{Z})) \otimes M_{\mathbf{k}_2}(\mathrm{Sp}_{n_2}(\mathbb{Z})),$$

whose precise definition will be given in Section 3. In this introduction, we focus on the symmetric case $n_1 = n_2 = n$.

Let us consider the scalar-valued case, that is, the case $l_1 = \dots = l_n$. If $k_0 = l_n$, this is just the restriction of the function on $\mathbb{H}_{2n}$ to $\mathbb{H}_n \times \mathbb{H}_n$,

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and the case $k_0 < l_n$ is more interesting. This type of differential operator was investigated by [4, 6, 9], and plays an important role in the arithmetic of modular forms. Indeed, by using this type of differential operator we can obtain the special value of the standard L-function at a non-rightmost critical point (cf. [19]), and give a criterion for the congruence of modular forms in terms of such standard L-values (cf. [18, 20]). Moreover, we can check Ikeda's conjecture on the period of the Ikeda-Miyawaki lift very smoothly (cf. [11]).

Now let us consider the case $k_0 = l-\nu$ and $\mathbf{k} = (\overbrace{k,\dots,k}^{m},\overbrace{l,\dots,l}^{n-m})$, $\boldsymbol{\ell} = \lambda^{(\nu)} := (\overbrace{k-l+\nu,\dots,k-l+\nu}^{m},\overbrace{\nu,\dots,\nu}^{n-m})$ with $k \geq l \geq l-\nu \geq 0$. In the case $\nu = 0$, an explicit formula for $\widetilde{\mathbb{D}}_{l,\lambda^{(0)},n,n}$ was given in some cases (cf. [2, 5, 10, 21]). This is also interesting and important. However, we remark that $\widetilde{\mathbb{D}}_{l,\lambda^{(0)},n,n}(M_l(\mathrm{Sp}_{2n}(\mathbb{Z})))$ is not contained in $S_{\mathbf{k}}(\mathrm{Sp}_n(\mathbb{Z})) \otimes S_{\mathbf{k}}(\mathrm{Sp}_n(\mathbb{Z}))$. Now we are interested in the case $\nu > 0$ and $k > l$, since it is expected to play the same role as in the scalar-valued case. Indeed, in [8], we gave an explicit formula for $\widetilde{\mathbb{D}}_{l-\nu,\lambda^{(\nu)},2,2}$ in the case $(k,l,\nu) = (14,10,2),(21,11,1),(21,11,3)$, and obtained the non-rightmost critical values of the standard L-function of vector-valued Siegel modular forms of genus 2 (see also [12]). We obtained this result by solving a system of linear equations derived from the pluriharmonicity of homogeneous polynomials, case by case. This method is very powerful, and in principle, given ν,λ,n concretely, we can obtain an explicit form of $\widetilde{\mathbb{D}}_{l-\nu,\lambda^{(\nu)},n,n}$ involving l. However, at present, it seems difficult to extract from this procedure a closed formula that applies uniformly to general n for given $l-\nu$ and $\lambda^{(\nu)}$.

On the other hand, Kozima [22] gave such a differential operator by another method. To be more precise, let $\mathcal{D}^{\nu}_{n,\alpha}$ be Böcherer's (unrestricted) differential operator (cf. (3.1)). Then Kozima showed that $\widetilde{\mathbb{D}}_{l-\nu,\lambda^{(\nu)},n,n}$ can be expressed as

$$\widetilde{\mathbb{D}}_{l-\nu,\lambda^{(\nu)},n,n} = \widetilde{\mathbb{D}}_{l,\lambda,n,n} \circ \mathcal{D}^{\nu}_{n,l-\nu} \text{ with } \lambda = (\overbrace{k-l,\dots,k-l}^{m},\overbrace{0,\dots,0}^{n-m}).$$

Strictly speaking, the proof was given only for $m = 1$, but the argument extends without difficulty to the general case. An explicit formula for $\widetilde{\mathbb{D}}_{l,\lambda,n,n}$ is well known as discussed before. Therefore it suffices to give an explicit description of the operator $\mathcal{D}^{\nu}_{n,\alpha}$.

In this paper, we treat the case $\nu = 2$ systematically and give an explicit formula for $\widetilde{\mathbb{D}}_{l-2,\lambda^{(2)},n,n}$ (cf. Theorem 3.13 and Corollary 3.14). As an application we give an algorithm for computing the special value of standard L-function at $s = l-n-2$ (the next rightmost critical point) of a Hecke eigenform in $S_{\mathbf{k}}(\mathrm{Sp}_n(\mathbb{Z}))$ (cf. Corollary 4.3, Proposition 4.4) and give such values in the case $n = 3$ (cf. Theorem 5.3). Moreover, we produce a congruence between cuspidal Hecke eigenforms (cf. Theorem 5.5). The proof of Theorem 3.13 relies on a careful analysis of several operators in [4].

For our purpose, it suffices to know $\mathcal{D}^{\nu}_{n,l-\nu}(\exp(2\pi\sqrt{-1}\,\mathrm{tr}(AW)))$ for a half-integral matrix A of size $2n$, where $W = (w_{ij})_{1\le i,j\le 2n} \in \mathbb{H}_{2n}$. After dividing by $\exp(2\pi\sqrt{-1}\,\mathrm{tr}(AW))$, this is a polynomial in $w_{i,j+n}$ $(1 \le i, j \le n)$ of degree νn with coefficients in $\mathbb{Q}$. Even for $\nu = 2$, the full polynomial is complicated; however, its reduction modulo $\mathcal{J}$ is sufficient for our application and is relatively simple (cf. (3.7), (3.8), and (3.9)), where $\mathcal{J}$ is the ideal of $\mathbb{Q}[w_{i,j+n}\ (1 \le i, j \le n)]$ generated by $w_{i_1,j_1+n}w_{i_2,j_2+n} - w_{i_1,j_2+n}w_{i_2,j_1+n}$ $(1 \le i_1, i_2, j_1, j_2 \le n)$.

We note that a more systematic method is given in [10]. Using it, for a general $\boldsymbol{\ell}$ and k_0, we can obtain an explicit formula for $\widetilde{\mathbb{D}}_{k_0,\boldsymbol{\ell},n,n}$ if one knows an explicit formula for a certain polynomial P_0 attached to $\boldsymbol{\ell}$ (cf. [10, Theorems 3 and 4]). It would be interesting to try to prove our formula using this approach.

This paper is organized as follows. In Section 2, we review a basic fact about Siegel modular forms and the Hecke theory for them. In Section 3, after reviewing the differential operators in [4, 6, 5, 22, 8], we give an explicit formula for $\widetilde{\mathbb{D}}_{l-2,\lambda^{(2)},n,n}$ (cf. Theorems 3.3 and 3.13). This section is the technical heart of this paper. In Section 4, we give a formula for the standard L-values based on the pullback formula for the Siegel-Eisenstein series (cf. Corollary 4.3). In Section 5, using the results in Sections 3 and 4, we give exact values of the standard L-functions at the next rightmost critical points of Hecke eigenforms of genus 3, and give examples of congruence between two cuspidal Hecke eigenforms.

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Notation. We denote by $\mathbb{Z}$, the ring of rational integers, and by $\mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ the fields of rational numbers, real numbers, and complex numbers, respectively. Moreover for a prime number p, we denote by $\mathbb{Q}_p$ and $\mathbb{Z}_p$ the field of p-adic numbers and the ring of p-adic integers, respectively. We also denote by ord_p the normalized additive valuation on $\mathbb{Q}_p$ and write $\mathbf{e}(z) = e^{2\pi\sqrt{-1}z}$ for $z \in \mathbb{C}$.

For a commutative ring R, we denote by $M_{mn}(R)$ the set of (m, n)-matrices with entries in R. In particular, put $M_n(R) = M_{nn}(R)$. For an (m, n)-matrix X and an (m, m)-matrix A, we write $A[X] = {}^tXAX$, where ${}^tX$ denotes the transpose of X. Put

$$\mathrm{GL}_m(R) = \{A \in M_m(R) \mid \det A \in R^\times\},$$

where $\det A$ denotes the determinant of the square matrix A, and $R^\times$ denotes the unit group of R. Let $S_n(R)$ denote the set of symmetric matrices of degree n with entries in R. Furthermore, for an integral domain R of characteristic different from 2, let $\mathcal{H}_n(R)$ denote the set of half-integral matrices of degree n over R, that is, $\mathcal{H}_n(R)$ is the set of symmetric matrices of degree n whose (i, j)-component belongs to R or $\frac{1}{2}R$ according as $i = j$ or not. If S is a subset of $\mathcal{H}_n(\mathbb{R})$ with $\mathbb{R}$ the field of real numbers, we denote by $S_{>0}$ (resp. $S_{\geq 0}$) the subset of S consisting of positive definite (resp. semi-positive definite) matrices. For square matrices X and Y we write $X \perp Y = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$.

2. Siegel modular forms and Hecke theory

In this section, we review basic facts about Siegel modular forms and the Hecke theory for them in [2, Section 2,3] with a little modification. We denote by $\mathbb{H}_n$ the Siegel upper half-space of degree n, i.e.,

$$\mathbb{H}_n = \{Z \in M_n(\mathbb{C}) \mid Z = {}^tZ = X + \sqrt{-1}Y,\ X, Y \in M_n(\mathbb{R}), Y > 0\}.$$

For any ring R and any positive integer n, we define the group $\mathrm{GSp}_n(R)$ by

$$\mathrm{GSp}_n(R) = \{g \in M_{2n}(R) \mid gJ_n{}^tg = \nu(g)J_n \text{ with some } \nu(g) \in R^\times\},$$

where $J_n = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$. We call $\nu(g)$ the symplectic similitude of g. We also define the symplectic group of degree n over R by

$$\mathrm{Sp}_n(R) = \{g \in \mathrm{GSp}_n(R) \mid \nu(g) = 1\}.$$

In particular, if R is a subfield of $\mathbb{R}$, we define

$$\mathrm{GSp}_n^+(R) = \{g \in \mathrm{GSp}_n(R) \mid \nu(g) > 0\}.$$

We put $\Gamma^{(n)} = \mathrm{Sp}_n(\mathbb{Z})$ for the sake of simplicity.

Let $\lambda = (k_1, k_2, \ldots)$ be a finite or an infinite sequence of non-negative integers such that $k_i \geq k_{i+1}$ for all i and $k_m = 0$ for some m. We call this a dominant integral weight. We call the biggest integer m such that $k_m \neq 0$ a depth of λ and write it by $\mathrm{depth}(\lambda)$. It is well known that the set of dominant integral weights λ with $\mathrm{depth}(\lambda) \leq n$ corresponds bijectively to the isomorphism classes of irreducible polynomial representations of $\mathrm{GL}_n(\mathbb{C})$. We denote this representation by $(\rho_{n,\lambda}, V_{n,\lambda})$. We also denote it by $(\rho_{\mathbf{k}}, V_{\mathbf{k}})$ with $\mathbf{k} = (k_1, \ldots, k_n)$ and call it an irreducible polynomial representation of $\mathrm{GL}_n(\mathbb{C})$ of highest weight $\mathbf{k}$. Moreover, we write $\mathbf{k}' = (k_1 - k_n, \ldots, k_{n-1} - k_n, 0)$. Then, we have

$\rho_{\mathbf{k}} \cong \det^{k_n} \otimes \rho_{\mathbf{k}'}$ with $(\rho_{\mathbf{k}'}, V_{\mathbf{k}'})$ an irreducible polynomial representation of highest weight $\mathbf{k}'$. Here we understand that $(\rho_{\mathbf{k}'}, V_{\mathbf{k}'})$ is the trivial representation on $\mathbb{C}$ if $k_1 = \cdots = k_{n-1} = k_n$. We fix a Hermitian inner product $\langle *, \ * \rangle$ on $V' = V_{\mathbf{k}'}$ such that

$$\langle \rho_{\mathbf{k}'}(g)v, w\rangle = \langle v, \rho_{\mathbf{k}'}({}^t\overline{g})w\rangle \quad \text{for } g \in \mathrm{GL}_n(\mathbb{C}), v, w \in V'.$$

Now we define vector-valued Siegel modular forms of $\Gamma^{(n)}$. For any V'-valued function F on $\mathbb{H}_n$, and for any $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{GSp}_n^+(\mathbb{R})$, we put $J(g, Z) = CZ + D$ and

$$F|_{\rho_{\mathbf{k}}}[g] = \rho_{\mathbf{k}}(J(g, Z))^{-1}F(gZ).$$

From now on we identify $\rho_{\mathbf{k}}$ with $\det^{k_n} \otimes \rho_{\mathbf{k}'}$. We say that F is a C^∞-modular form of weight $\rho_{\mathbf{k}}$ or $\mathbf{k}$ with respect to $\Gamma^{(n)}$ if F is a C^∞-mapping from $\mathbb{H}_n$ to V' satisfying the following condition:

$$F|_{\rho_{\mathbf{k}}}[\gamma] = F \text{ for any } \gamma \in \Gamma^{(n)}.$$

We denote by $M_{\mathbf{k}}^\infty(\Gamma^{(n)}) = M_{\rho_{\mathbf{k}}}^\infty(\Gamma^{(n)})$ the space of C^∞-modular forms of weight $\rho_{\mathbf{k}}$ with respect to $\Gamma^{(n)}$. We say that an element F of $M_{\mathbf{k}}^\infty(\Gamma^{(n)})$ is a (holomorphic) modular form of weight $\rho_{\mathbf{k}}$ if F is a holomorphic mapping from $\mathbb{H}_n$ to V' and has the following Fourier expansion

$$F(Z) = \sum_{T \in \mathcal{H}_n(\mathbb{Z})_{\geq 0}} a(T, F)\mathbf{e}(\mathrm{tr}(TZ)), \quad Z \in \mathbb{H}_n,\ a(T, F) \in V',$$

where $\mathrm{tr}(T)$ is the trace of a matrix T. We note that F has the above Fourier expansion automatically if $n \geq 2$. We denote by $M_{\mathbf{k}}(\Gamma^{(n)}) = M_{\rho_{\mathbf{k}}}(\Gamma^{(n)})$ the space of modular forms of weight $\rho_{\mathbf{k}}$ with respect to $\Gamma^{(n)}$. We say that $F \in M_{\rho_{\mathbf{k}}}(\Gamma^{(n)})$ is a cusp form if we have $a(T, F) = 0$ unless T is positive definite. We denote by $S_{\mathbf{k}}(\Gamma^{(n)}) = S_{\rho_{\mathbf{k}}}(\Gamma^{(n)})$ the subspace of $M_{\rho_{\mathbf{k}}}(\Gamma^{(n)})$ consisting of cusp forms.

For $F, G \in M_{\rho_{\mathbf{k}}}^\infty(\Gamma^{(n)})$ the Petersson inner product is defined by

$$(F, G) = \int_{\Gamma^{(n)}\backslash \mathbb{H}_n} \langle \rho_{\mathbf{k}}(\sqrt{Y})F(Z), \rho_{\mathbf{k}}(\sqrt{Y})G(Z)\rangle \det(Y)^{-n-1}dZ,$$

where $Y = \mathrm{Im}(Z)$ and $\sqrt{Y}$ is a positive definite symmetric matrix such that $\sqrt{Y}^2 = Y$. This integral converges if F and G are slowly increasing and at least one of them belongs to $S_{\rho_{\mathbf{k}}}(\Gamma^{(n)})$. If $\mathbf{k} = (\overbrace{k, \ldots, k}^{n})$, we simply write $M_k(\Gamma^{(n)}) = M_{\mathbf{k}}(\Gamma^{(n)})$ and $S_k(\Gamma^{(n)}) = S_{\mathbf{k}}(\Gamma^{(n)})$.

Now let $\boldsymbol{\ell} = (l_1, \ldots, l_n)$ be a dominant integral weight of length n of depth m. Then we can realize the representation space $V_{\boldsymbol{\ell}}$ in terms of bideterminants. Let $\mathcal{BD}_{\boldsymbol{\ell}}$ be the set of all bideterminants of weight $\boldsymbol{\ell}$.

For a commutative ring R and an R-algebra S let $S[U]_{\boldsymbol{\ell}}$ denote the R-module of all S-linear combinations of $P(U)$ for $P(U) \in \mathcal{BD}_{\boldsymbol{\ell}}$. For the definition of bideterminants, see [2], [14] in general case.

Here, for our later purpose, we explain the case $\boldsymbol{\ell} = (l, 0, \ldots, 0)$ precisely. In this case $\mathcal{BD}_{\boldsymbol{\ell}}$ is the set of monomials $u_1^{a_1} \cdots u_n^{a_n}$ of degree l, and $S[U]_{\boldsymbol{\ell}}$ is the R-module of homogeneous polynomials in $u_1, \ldots, u_n$ of degree l with coefficients in S. Let $\widetilde{V} = \widetilde{V}_{\boldsymbol{\ell}} = \mathbb{Q}[U]_{\boldsymbol{\ell}}$. Then, $(\rho_{\boldsymbol{\ell}}|\mathrm{GL}_n(\mathbb{Q}), \widetilde{V})$ is a representation of $\mathrm{GL}_n(\mathbb{Q})$, and $\widetilde{V} \otimes \mathbb{C} = V_{\boldsymbol{\ell}}$. We consider a $\mathbb{Z}$-structure of $V_{\boldsymbol{\ell}}$. To do this, we fix a basis $\mathcal{S} = \mathcal{S}_{\boldsymbol{\ell}}$ of $\mathbb{Z}[U]_{\boldsymbol{\ell}}$. For example, in the case $\boldsymbol{\ell} = (l, 0, \ldots, 0)$, the set of monomials of degree l can be taken as a basis of $\mathbb{Z}[U]_{\boldsymbol{\ell}}$. We note here that the bideterminants are not linearly independent over $\mathbb{C}$ and even over $\mathbb{Z}$ in general, so the set $\mathcal{BD}_{\boldsymbol{\ell}}$ is not necessarily a basis of $\mathbb{Z}[U]_{\boldsymbol{\ell}}$. Let R be a subring of $\mathbb{C}$. Since the set $\mathcal{S}$ is also linearly independent over $\mathbb{C}$, an element a of $R[U]_{\boldsymbol{\ell}}$ is uniquely written as

$$a = \sum_{P \in \mathcal{S}} a_P P \text{ with } a_P \in R.$$

Let K be a number field, and $\mathfrak{O}$ the ring of integers in K. For a prime ideal $\mathfrak{p}$ of $\mathfrak{O}$ and $a = a(U) = \sum_{P \in \mathcal{S}} a_P P \in K[U]_{\boldsymbol{\ell}}$ with $a_P \in K$, define

$$\mathrm{ord}_{\mathfrak{p}}(a) = \min_{P \in \mathcal{S}} \mathrm{ord}_{\mathfrak{p}}(a_P).$$

We say that $\mathfrak{p}$ divides a if $\mathrm{ord}_{\mathfrak{p}}(a) > 0$. For a subring R of $\mathbb{C}$, we denote by $M_{\mathbf{k}}(\Gamma^{(n)})(R)$ the R-submodule of $M_{\mathbf{k}}(\Gamma^{(n)})$ consisting of all modular forms F such that $a(T, F) \in R[U]_{\mathbf{k}'}$ for all $T \in \mathcal{H}_n(\mathbb{Z})_{\geq 0}$. Here, $\mathbf{k}' = (k_1 - k_{m+1}, \ldots, k_m - k_{m+1}, \overbrace{0, \ldots, 0}^{n-m})$ for $\mathbf{k} = (k_1, \ldots, k_n)$ with $k_1 \geq \cdots \geq k_m > k_{m+1} = \cdots = k_n$ as stated before. We can define tensor products of modular forms, which will be used on and after Section 4 . For the details, see [2, Section 2].

Now we review the Hecke theory for Siegel modular forms following [2]. Let $\mathbf{L}_n = \mathbf{L}(\Gamma^{(n)}, \mathrm{GSp}_n^+(\mathbb{Q}) \cap M_{2n}(\mathbb{Z}))$ be the Hecke ring over $\mathbb{Z}$ for the Hecke pair $(\Gamma^{(n)}, \mathrm{GSp}_n^+(\mathbb{Q}) \cap M_{2n}(\mathbb{Z}))$, and $\widetilde{\mathbf{L}}_n = \mathbf{L}(\Gamma^{(n)}, \mathrm{GSp}_n^+(\mathbb{Q}))$ be the Hecke ring over $\mathbb{Q}$ for the Hecke pair $(\Gamma^{(n)}, \mathrm{GSp}_n^+(\mathbb{Q}))$. As for the definition of the Hecke pairs and the Hecke rings, see [1, 3.1.2]. For $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ with $k_1 \geq \cdots \geq k_n \geq 0$ and $F \in M_{\mathbf{k}}(\Gamma^{(n)})$ the action of T on F is defined as

$$F|T = \nu(g)^{k_1 + \cdots + k_n - n(n+1)/2} \sum_g F|_{\rho_{\mathbf{k}}} g,$$

and call it the Hecke operator. From now on, we identify T with the operator $F \mapsto F|T$ on $M_{\mathbf{k}}(\Gamma^{(n)})$. We define the elements $T(p)$ and

$T_j(p^2)$ of $\mathbf{L}_n$ in a standard way (cf. [1]). We say that a modular form $F \in M_{\mathbf{k}}(\Gamma^{(n)})$ is a Hecke eigenform if F is a common eigenfunction of all $T \in \widetilde{\mathbf{L}}_n$. In this case, we denote by $\mathbb{Q}(F)$ the field generated over $\mathbb{Q}$ by all the Hecke eigenvalues of $T \in \widetilde{\mathbf{L}}_n$, and call it the Hecke field of F. We say that an element $T \in \widetilde{\mathbf{L}}_n$ is integral with respect to $M_{\mathbf{k}}(\Gamma^{(n)})$ if $F|T \in M_{\mathbf{k}}(\Gamma^{(n)})(\mathbb{Z})$ for any $F \in M_{\mathbf{k}}(\Gamma^{(n)})(\mathbb{Z})$. We denote by $\mathbf{L}_n^{(\mathbf{k})}$ the subset of $\widetilde{\mathbf{L}}_n$ consisting of all integral elements with respect to $M_{\mathbf{k}}(\Gamma^{(n)})$. From now on assume that $k_n \geq n+1$. Then we have $\mathbf{L}_n \subset \mathbf{L}_n^{(\mathbf{k})}$ and for any Hecke eigenform $F \in M_{\mathbf{k}}(\Gamma^{(n)})$ and $T \in \mathbf{L}_n^{(\mathbf{k})}$, $\lambda_F(T)$ is an algebraic integer (cf. [2, Propositions 3.1 and 3.2]).

3. Differential operators

In this section, first we review the differential operator on modular forms following [4, 6, 5, 10, 22], and next we give an explicit formula for the differential operator which sends $M_{l-2}(\Gamma^{(2n)})$ to $S_{\mathbf{k}}(\Gamma^{(n)}) \otimes S_{\mathbf{k}}(\Gamma^{(n)})$ with $\mathbf{k} = (k, \overbrace{l, \ldots, l}^{n-1})$.

Let

$$\mathcal{I}_{n,r} = \{\mathbf{i} = (i_1, \ldots, i_r) \in \mathbb{Z}^r \mid 1 \leq i_1 < \cdots < i_r \leq n\},$$

and we define the lexicographic order on $\mathcal{I}_{n,r}$ in a usual way, that is, for $\mathbf{i} = (i_1, \ldots, i_r)$ and $\mathbf{j} = (j_1, \ldots, j_r) \in \mathcal{I}_{n,r}$ we write $\mathbf{i} \succ \mathbf{j}$ if there is an integer $1 \leq s \leq r+1$ such that $i_1 = j_1, \cdots, i_{s-1} = j_{s-1}$ and $i_s > j_s$. Here we make the convention that $i_1 > j_1$ if $s = 1$ and that $i_1 = j_1, \cdots, i_r = j_r$ if $s = r+1$. For an element $\mathbf{i} = (i_1, \ldots, i_r) \in \mathbb{Z}^r$ we denote by $\mathrm{supp}(\mathbf{i}) = \{i_1, \ldots, i_r\}$ and by $|\mathbf{i}| = i_1 + \cdots + i_r$. For $\mathbf{i} \in \mathbb{Z}^r, \mathbf{j} = \mathbb{Z}^s$, we write $u(\mathbf{i}, \mathbf{j}) = \#(\mathrm{supp}(\mathbf{i}) \cup \mathrm{supp}(\mathbf{j}))$ and $c(\mathbf{i}, \mathbf{j}) = \#(\mathrm{supp}(\mathbf{i}) \cap \mathrm{supp}(\mathbf{j}))$. For $\mathbf{i} \in \mathcal{I}_{n,r}, \mathbf{j} = \mathcal{I}_{n,s}$ we denote by $\mathbf{i} \cup \mathbf{j}$ (resp. $\mathbf{i} \cap \mathbf{j}$) the element of $\mathcal{I}_{n,u(\mathbf{i},\mathbf{j})}$ (resp. $\mathcal{I}_{n,u(\mathbf{i},\mathbf{j})}$) such that $\mathrm{supp}(\mathbf{i} \cup \mathbf{j}) = \mathrm{supp}(\mathbf{i}) \cup \mathrm{supp}(\mathbf{j})$ (resp. $\mathrm{supp}(\mathbf{i} \cap \mathbf{j}) = \mathrm{supp}(\mathbf{i}) \cap \mathrm{supp}(\mathbf{j})$). Here we write $\mathbf{i} \cap \mathbf{j} = \emptyset$ if $\mathrm{supp}(\mathbf{i}) \cap \mathrm{supp}(\mathbf{j}) = \emptyset$. If $\mathbf{i} \cap \mathbf{j} = \emptyset$, $\mathbf{i} \cup \mathbf{j}$ is written as $\mathbf{i} \sqcup \mathbf{j}$. For an element $\mathbf{i} \in \mathcal{I}_{n,r}$ and $\mathbf{a} \in \mathcal{I}_{n,s}$ such that $\mathrm{supp}(\mathbf{a}) \subset \mathrm{supp}(\mathbf{i})$, we denote by $\mathbf{i} \backslash \mathbf{a}$ the element of $\mathcal{I}_{n,r-s}$ such that $\mathrm{supp}(\mathbf{i} \backslash \mathbf{a}) \cup \mathrm{supp}(\mathbf{a}) = \mathrm{supp}(\mathbf{i})$. In particular, we write $\mathbf{i}^c = (1, \ldots, n) \setminus \mathbf{i}$ for $\mathbf{i} \in \mathcal{I}_{n,r}$. Let $(k_1, \ldots, k_{r+s}) = \mathbf{a} \sqcup \mathbf{b}$ with $\mathbf{a} = (i_1, \ldots, i_r) \in \mathcal{I}_{n,r}, \mathbf{b} = (j_1, \ldots, j_s) \in \mathcal{I}_{n,s}$. Then we denote by $\epsilon(\mathbf{a}, \mathbf{b})$ the sign of the permutation σ in $r+s$ letters such that

$$\sigma(k_a) = \begin{cases} i_a & \text{for } 1 \leq a \leq r \\ j_{a-r} & \text{for } r+1 \leq a \leq r+s \end{cases}.$$

For a commutative ring R we define

$$M_{\mathcal{I}_{m,p},\mathcal{I}_{n,q}}(R) = \{(a_{st})_{s\in\mathcal{I}_{m,p},\ t\in\mathcal{I}_{n,q}}\ a_{s,t}\in R\},$$

and in particular,

$$M_{\mathcal{I}_{n,p}}(R) = M_{\mathcal{I}_{n,p},\mathcal{I}_{n,p}}(R).$$

We note that $M_{\mathcal{I}_{m,1},\mathcal{I}_{n,1}}(R)$ is the set of $m\times n$ matrices $M_{m,n}(R)$ with entries in R. For $A=(a_{s,t})_{s,t\in\mathcal{I}_{n,p}}\in M_{\mathcal{I}_{n,p}}(R), B=(b_{s',t'})_{s',t'\in\mathcal{I}_{n,q}}\in M_{\mathcal{I}_{n,q}}(R)$ we define an element $A\sqcap B$ of $M_{\mathcal{I}_{n,p+q}}(R)$ as

$$A\sqcap B = (c_{s,t})_{s,t\in\mathcal{I}_{n,p+q}},$$

where

$$c_{s,t} = \frac{1}{\binom{p+q}{p}}\sum_{s'\sqcup t'=s, t'\sqcup t''=t}\varepsilon(s',s'')a_{s',t'}b_{s'',t''}.$$

For an $n\times n$ matrix $A=(a_{ij})$ and $\mathbf{i}=(i_1,\ldots,i_r), \mathbf{j}=(j_1,\ldots,j_r)\in\mathcal{I}_{n,r}$ put $A\begin{pmatrix}\mathbf{i}\\ \mathbf{j}\end{pmatrix}=\det\left(a_{i_a,j_b}\right)_{1\le a,b\le r}$, and

$$A^{[r]} = \left(A\begin{pmatrix}\mathbf{i}\\ \mathbf{j}\end{pmatrix}\right)_{\mathbf{i},\mathbf{j}\in\mathcal{I}_{n,r}}.$$

We sometimes write $A\begin{pmatrix}i_1,\ldots,i_r\\ j_1,\ldots,j_r\end{pmatrix}$ instead of $A\begin{pmatrix}(i_1,\ldots,i_r)\\ (j_1,\ldots,j_r)\end{pmatrix}$. We note that

$$A^{[r]} = \overbrace{A\sqcup\cdots\sqcup A}^{r}.$$

We also define $\mathrm{Ad}^{[r]}A$ as

$$\mathrm{Ad}^{[r]}A = \left(\varepsilon(\mathbf{i},\mathbf{i}^c)\varepsilon(\mathbf{j},\mathbf{j}^c)A\begin{pmatrix}\mathbf{i}^c\\ \mathbf{j}^c\end{pmatrix}\right)_{\mathbf{i},\mathbf{j}\in\mathcal{I}_{n,r}}$$

Moreover put $E_{\mathbf{i},\mathbf{j}}=\left(\delta_{\mathbf{i},\mathbf{a}}\delta_{\mathbf{j},\mathbf{b}}\right)_{\mathbf{a},\mathbf{b}\in\mathcal{I}_{n,r}}$, and in particular, put $E_{ij}=\left(\delta_{i,a}\delta_{j,a}\right)_{1\le a,b\le n}$. Here $\delta_{\mathbf{i},\mathbf{a}}=1$ or 0 according as $\mathbf{a}=\mathbf{i}$ or not, and the others. We note that $A^{[0]}=1$, $A^{[n]}=\det(A)$, $\mathrm{Ad}^{[n]}A=1$ and that if $0<r<n$, then $A^{[r]}$ is a matrix such that each component is a homogeneous polynomial of components of A of positive order. As for precise properties of these operators see [4].

Let $W=(w_{ij})_{1\le i,j\le 2n}$ be a matrix of variables with $w_{ij}=w_{ji}$, and we write $\partial_{ij}=\frac{\widetilde{\partial}}{\partial w_{ij}}=\frac{(1+\delta_{ij})}{2}\frac{\partial}{\partial w_{ij}}$, and $\left(\frac{\partial}{\partial W}\right)=(\partial_{ij})_{1\le i,j\le 2n}$. Write $W=\begin{pmatrix}Z_1 & Z_2\\ {}^tZ_2 & Z_4\end{pmatrix}\in\mathbb{H}_{2n}$, and put $\partial_1=\frac{\partial}{\partial Z_1}=(\partial_{ij})_{1\le i,j\le n}$, $\partial_2=\frac{\partial}{\partial Z_2}=(\partial_{i,j+n})_{1\le i,j\le n}$, and $\partial_4=\frac{\partial}{\partial Z_4}=(\partial_{i+n,j+n})_{1\le i,j\le n}$ We sometimes write $Z=Z_2$ and $\partial_2=\frac{\partial}{\partial Z}$, and denote by z_{ij} the (i,j)-entry of Z, where $1\le i,j\le n$.

We use the notation in [4] or [6] and we put

$$\Delta(r,q)=\sum_{a+b=q}(-1)^b\binom{q}{b}Z^{[a]}\partial_4^{[a]}\sqcap(1_n^{[r]}\sqcap Z^{[b]}({}^t\partial_2)^{[b]})(\mathrm{Ad}^{[r+b]}\partial_1)\partial_2^{[r+b]},$$

and following [6], we define

$$\mathcal{D}_{n,\alpha}=\sum_{r+q=n}(-1)^r\binom{n}{r}C_r(\alpha-n+1/2)\Delta(r,q).$$

For non-negative integers ν and α, we define $\mathcal{D}^\nu_{n,\alpha}$ as

$$\mathcal{D}^\nu_{n,\alpha}=\mathcal{D}_{n,\alpha+\nu-1}\circ\cdots\mathcal{D}_{n,\alpha+1}\circ\mathcal{D}_{n,\alpha}. \tag{3.1}$$

Remark 3.1. The definition of $\mathcal{D}_{n,\alpha}$ and $\mathcal{D}^\nu_{n,\alpha}$ are different from those in [4] or [8]. More precisely,

$$\mathcal{D}_{n,\alpha}=(-1)^nC_n(\alpha-n+1/2)\widetilde{\mathcal{D}}_\alpha$$

and therefore

$$\mathcal{D}^\nu_{n,\alpha}=(-1)^{n\nu}\prod_{i=0}^{\nu-1}C_n(\alpha-n+i+1/2)\widetilde{\mathcal{D}}^\nu_\alpha,$$

where $\widetilde{\mathcal{D}}_\alpha$ and $\mathcal{D}^\nu_\alpha$ are the differential operators in [8, Section 7].

The operator $\mathcal{D}^\nu_{n,\alpha}$ maps $C^\infty(\mathbb{H}_{2n},\mathbb{C})$ to $C^\infty(\mathbb{H}_{2n},\mathbb{C})$ for any non-negative integer ν. Let T be an $m\times n$ matrix of variables. Then we say that $P(T)$ is a polynomial in T if it is a polynomial in t_{ij} ($1\le i\le m, 1\le j\le n$). In particular, let $T=(t_{ij})_{1\le i,j\le m}$ be a symmetric matrix of variables. We say that $P(T)$ is a polynomial in T if it is a polynomial in t_{ij} ($1\le i\le j\le m$).

We note that we have

(3.2)

$$\partial_i^{[r]}(\exp(\mathrm{tr}(S_iZ_i)))=S_i^{[r]}\exp(\mathrm{tr}(S_iZ_i)) \text{ for } S_i\in S_n(\mathbb{C}) \text{ with } i=1,4$$

and

$$\partial_2^{[r]}(\exp(\mathrm{tr}(S_2Z)))=(S_2/2)^{[r]}\exp(\mathrm{tr}(S_2Z)) \text{ for } S_2\in M_n(\mathbb{C}). \tag{3.3}$$

In particular, for $S=\begin{pmatrix}S_1 & S_2\\ {}^tS_2 & S_4\end{pmatrix}$

$$\partial_i^{[r]}(\exp(\mathrm{tr}(SW)))=S_i^{[r]}\exp(\mathrm{tr}(SW)) \tag{3.4}$$

Therefore, for $f(W)=\exp(\mathrm{tr}(TW))$ with $W=\begin{pmatrix} Z_1 & Z \\ {}^tZ & Z_4 \end{pmatrix}\in \mathbb{H}_{2n}, T\in S_{2n}(\mathbb{R})$, $\mathcal{D}_{n,\alpha}^{\nu}(f)$ can be expressed as

$$\mathcal{D}_{n,\alpha}^{\nu}(f)=\sum_{i=0}^{\nu n} P_{i,\nu,n,\alpha}(T,Z)f, \tag{3.5}$$

where $P_{i,\nu,n,\alpha}(T,Z)$ is a homogeneous polynomial in Z of degree i with coefficients in $\mathbb{Q}$. Therefore, for $f(W)=\exp(2\pi\sqrt{-1}\,\mathrm{tr}(TW))$, we have

$$\mathcal{D}_{n,\alpha}^{\nu}(f)=(2\pi\sqrt{-1})^{\nu n}\sum_{i=0}^{\nu n} P_{i,\nu,n,\alpha}(T,Z)f.$$

Let $\boldsymbol{\ell}=(l_1,l_2,\ldots)$ be a finite or infinite sequence of non-negative integers of depth m. For integers $n_1,n_2\geq m$ and a positive integer $k_0\geq \max(n_1,n_2)$, put $\mathbf{k}_1=(l_1+k_0,\ldots,l_{n_1}+k_0), \mathbf{k}_2=(l_1+k_0,\ldots,l_{n_2}+k_0)$. Then by [10, Theorems 3 and 4], there exists a unique differential operator, up to a constant multiple, that sends

$$M_{k_0}(\varGamma^{(n_1+n_2)})\to M_{\mathbf{k}_1}(\varGamma^{(n_1)})\otimes M_{\mathbf{k}_2}(\varGamma^{(n_2)}).$$

We denote it by $\widetilde{\mathbb{D}}_{k_0,\boldsymbol{\ell},n_1,n_2}$. It is investigated by [10] in the general case.

In this paper we restrict ourselves to the case $n_1=n_2=n, k_0=l-\nu, \boldsymbol{\ell}=\lambda^{(\nu)}:=(\overbrace{k-l+\nu,\ldots,k-l+\nu}^{m},\overbrace{\nu,\ldots,\nu}^{n-m})$ with $k\geq l$ and put $\mathbf{k}_i=(\overbrace{k,\ldots,k}^{m},\overbrace{l,\ldots,l}^{n_i-m})$ for $i=1,2$. In particular, we put $\mathbf{k}=(\overbrace{k,\ldots,k}^{m},\overbrace{l,\ldots,l}^{n-m})$ if $n_1=n_2=n$.

Let $\lambda=(\overbrace{k-l,\ldots,k-l}^{m},\overbrace{0,\ldots,0}^{n-m})$ and $(\rho_{n_i,\lambda},V_{n_i,\lambda})$ be the polynomial representation of highest weight λ as in Section 2, and put $V_{\lambda,n_1,n_2}=V_{n_1,\lambda}\otimes V_{n_2,\lambda}$. Let $\mathbb{D}_{l,\lambda,n_1,n_2}$ be the differential operator in [2, Proposition 5.2]. In [2], this operator is denoted by $\mathbb{D}_{\lambda,n_1,n_2}$, but actually depends on (k,l,n_1,n_2).

The differential operator $\mathbb{D}_{l,\lambda,n_1,n_2}$ is a mapping from $C^\infty(\mathbb{H}_{n_1+n_2},\mathbb{C})$ to $C^\infty(\mathbb{H}_{n_1+n_2},V_{\lambda,n_1,n_2})$. The restriction of $\mathbb{D}_{l,\lambda,n_1,n_2}$ to $\mathbb{H}_{n_1}\times\mathbb{H}_{n_2}$ sends $M_l(\varGamma^{(n_1+n_2)})$ to $M_{\mathbf{k}_1}(\varGamma^{(n_1)})\otimes M_{\mathbf{k}_2}(\varGamma^{(n_2)})$, and in particular, it sends $S_l(\varGamma^{(n_1+n_2)})$ to $S_{\mathbf{k}_1}(\varGamma^{(n_1)})\otimes S_{\mathbf{k}_2}(\varGamma^{(n_2)})$. Therefore we can write it as $\widetilde{\mathbb{D}}_{l,\lambda,n_1,n_2}$. Let $\mathcal{D}_{n,\alpha}^{\nu}$ be the differential operator defined in [4] which maps $C^\infty(\mathbb{H}_{2n},\mathbb{C})$ to itself. Then, as stated in Section 1, for a non-negative integer ν, we can take $\widetilde{\mathbb{D}}_{l-\nu,\lambda^{(\nu)},n,n}$ as follows:

$$\widetilde{\mathbb{D}}_{l-\nu,\lambda^{(\nu)},n,n}(f)=\widetilde{\mathbb{D}}_{l,\lambda,n,n}\circ\mathcal{D}_{n,l-\nu}^{\nu}(f). \tag{3.6}$$

In particular, its image is contained in $S_{\mathbf{k}}(\Gamma^{(n)}) \otimes S_{\mathbf{k}}(\Gamma^{(n)})$ if $\nu > 0$. This can be easily proved by [4, (29b), (31)].

Throughout the rest of this paper, for integers $k \geq l \geq 0$ and $\nu \geq 0$, we put $\lambda = (k-l, 0, \dots, 0)$ and $\lambda^{(\nu)} = (k-l+\nu, \nu, \dots, \nu)$. Let $W = \begin{pmatrix} Z_1 & Z \\ {}^tZ & Z_4 \end{pmatrix} \in \mathbb{H}_{2n}$, where $Z_1 \in \mathbb{H}_n, Z_4 \in \mathbb{H}_n$ and $Z \in M_n(\mathbb{C})$. As stated in Section 2, we realize $V_{n,\lambda} \otimes V_{n,\lambda}$ as $\mathbb{C}[u]_\lambda \otimes \mathbb{C}[v]_\lambda$ with $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$. Then, as $\widetilde{\mathbb{D}}_{l,\lambda,n,n}$ we can take the following differential operator:

$$\begin{aligned}\widetilde{\mathbb{D}}_{l,\lambda,n,n} = \Big(&((k-2)!)^{-1} \\ &\times \sum_{\mu=0}^{(k-l)/2} (-1)^\mu \frac{(k-\mu-2)!}{(k-l-2\mu)!\mu!} (\frac{\partial}{\partial Z_1}[{}^tu] \frac{\partial}{\partial Z_4}[{}^tv])^\mu \\ &\times (u \frac{\partial}{\partial Z} {}^tv)^{k-l-2\mu} \Big)\Big|_{Z=O}.\end{aligned}$$

Remark 3.2. We note that this coincides with $(l)_{k-l} \widetilde{L}^{l,k-l}\big|_{Z=O}$, where $(l)_m = \prod_{i=1}^m (l+i-1)$ and $\widetilde{L}^{l,k-l}$ is the differential operator defined in [8, Page 1313]. (See also [5, (2.7)].)

Then the following formula holds.

Theorem 3.3. *Let* $W = \begin{pmatrix} Z_1 & Z \\ {}^tZ & Z_4 \end{pmatrix} \in \mathbb{H}_{2n}$ *be as above. For any* $T = \begin{pmatrix} A & R/2 \\ {}^tR/2 & B \end{pmatrix} \in S_{2n}(\mathbb{R})$ *put* $f(W) = \mathbf{e}(\mathrm{tr}(TW))$

$$\begin{aligned} Q_{l-\nu,\lambda^{(\nu)},n,n}(T; u, v) &= ((k-2)!)^{-1} \\ &\times \sum_{i=0}^{\nu n} \sum_{\mu=0}^{(k-l)/2} (-1)^\mu \frac{(k-\mu-2)!}{(k-l-2\mu-i)!\mu!} (A[{}^tu] B[{}^tv])^\mu \\ &\times (uR{}^tv)^{k-l-2\mu-i} P_{i,\nu,n,l-\nu}(T, (u_a v_b)_{1\leq a,b\leq n}).\end{aligned}$$

Then we have

$$\widetilde{\mathbb{D}}_{l,\lambda,n,n} \circ \mathcal{D}^\nu_{n,l-\nu}(f) = (2\pi\sqrt{-1})^{k-l+\nu n} Q_{l-\nu,\lambda^{(\nu)},n,n}(T; u, v) \mathbf{e}(\mathrm{tr}(AZ_1 + BZ_4)).$$

Here we make the convention that $({}^tuRv)^\nu = 0$ *if* $\nu < 0$.

Proof. The assertion follows directly from (3.5) and the following lemma. □

Lemma 3.4. *Let the notation be as in Theorem 3.3. Put*

$$\partial_Z(u, v) = \sum_{1 \leq i,j \leq n} v_i u_j \frac{\partial}{\partial z_{ij}}.$$

Moreover, for a positive integer m_0 and non-negative integers e_{ij} $(1 \le i, j \le n)$ such that $\sum_{1\le i,j\le n} e_{ij} = m_0$, put $\mathcal{E}_{m_0}(Z) = \prod_{1\le i,j\le n} z_{ij}^{e_{i,j}}$. Then for any non-negative integer m we have

$$\partial_Z(u,v)^m\Big(\mathcal{E}_{m_0}(Z)f(W)\Big)|_{Z=O} = (2\pi\sqrt{-1})^{m-m_0}$$

$$\times \begin{cases} \frac{m!}{(m-m_0)!}(uR^tv)^{m-m_0}\prod_{1\le i,j\le n}(v_iu_j)^{e_{ij}}\mathbf{e}(\mathrm{tr}(AZ_1+BZ_4)) & \textit{if } m \ge m_0 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. We have

$$\partial_Z(u,v)^m\Big(\mathcal{E}_{m_0}(Z)f(W)\Big) = \sum_{r=0}^{m}\binom{m}{r}\partial_Z(u,v)^r(\mathcal{E}_{m_0}(Z))\partial_Z(u,v)^{m-r}(f).$$

We easily see that

$$\partial_Z(u,v)^r(\mathcal{E}_{m_0}(Z))\Big|_{Z=O} = \begin{cases} m_0!\prod_{1\le i,j\le n}(v_iu_j)^{e_{ij}} & \text{if } r = m_0 \\ 0 & \text{otherwise} \end{cases}$$

Thus the assertion holds. □

From now on we mainly consider the case $\nu = 2$, and write $P_{i,\alpha}(T,Z) = P_{i,2,n,\alpha}(T,Z)$ if there is no fear of confusion. Let Z be an $n\times n$ matrix of variables. Then, for $T \in S_{2n}(\mathbb{R})$, there is a polynomial $\Delta(r, n-r, T) = \Delta(r, n-r, T)(Z)$ in Z such that

$$\begin{aligned} &\Delta(r, n-r)(\exp(\mathrm{tr}(TW))) \\ =&\Delta(r, n-r, T)\exp(\mathrm{tr}(TW)) = \Delta(r, n-r, T)(Z)\exp(\mathrm{tr}(TW)), \end{aligned}$$

where $W = \begin{pmatrix} Z_1 & Z \\ {}^tZ & Z_4 \end{pmatrix} \in \mathbb{H}_{2n}$ with $Z_1, Z_4 \in \mathbb{H}_n$.

To give an explicit formula for $\mathcal{D}^2_{3,\alpha}(f)$, we recall the following lemma (cf. [4, (15)].

Lemma 3.5. *Let $W = \begin{pmatrix} Z_1 & Z \\ {}^tZ & Z_4 \end{pmatrix} \in \mathbb{H}_{2n}$ with $Z_1, Z_4 \in \mathbb{H}_n$. Let $g(W) = \exp(\mathrm{tr}(AZ_1))\,h(Z,Z_4)$ with $A \in S_n(\mathbb{R})_{>0}$ and $h(Z,Z_4)$ a holomorphic function of Z and Z_4. Then we have*

$$\Delta(r, n-r)(g) = \big(AZ(\partial_4 - {}^t\partial_2A^{-1}\partial_2)^{[n-r]} \sqcap \partial_2^{[r]}\big)(g).$$

In particular, for $T = \begin{pmatrix} A & R/2 \\ {}^tR/2 & B \end{pmatrix} \in S_{2n}(\mathbb{R})$ with $A \in S_n(\mathbb{R})_{>0}$, we have

$$\Delta(r, n-r, T) = AZ(B - {}^t(R/2)A^{-1}(R/2))^{[n-r]} \sqcap (R/2)^{[r]}.$$

Theorem 3.6. *Let $f(W) = \exp(\mathrm{tr}(TW))$ with $T \in S_{2n}(\mathbb{R})$. Then*

$$\begin{aligned}\mathcal{D}^2_{n,\alpha}(f) = &\sum_{r=0}^{n}\sum_{s=0}^{n}(-1)^{r+s}\binom{n}{r}\binom{n}{s}\\ &\times C_r(\alpha+1-n+1/2)C_s(\alpha-n+1/2)\\ &\times \Delta(r,n-r)\big(\Delta(s,n-s,T)f\big)\end{aligned}$$

Lemma 3.7. (1) *Let $A = (a_{\mathbf{i},\mathbf{j}})_{\mathbf{i},\mathbf{j}\in\mathcal{I}_{n,r}}$ and $B = (b_{\mathbf{i},\mathbf{j}})_{\mathbf{i},\mathbf{j}\in\mathcal{I}_{n,n-r}}$. Then*

$$A \sqcap B = \binom{n}{n-r}^{-1}\sum_{\mathbf{i},\mathbf{j}\in\mathcal{I}_{n,r}}(-1)^{|\mathbf{i}|+|\mathbf{j}|}a_{\mathbf{i},\mathbf{j}}b_{\mathbf{i}^c,\mathbf{j}^c}.$$

(2) *Let r, s, t be non-negative integers such that $r+s+t=n$. Let A_1, A_2, B, C and Z be $n\times n$ matrices. Then*

$$(A_1ZA_2)^{[r]} \sqcap B^{[s]} \sqcap C^{[t]} = Z^{[r]} \sqcap ({}^t\mathrm{Ad}^{[n-r]}A_1(B^{[s]}\sqcap C^{[t]}){}^t\mathrm{Ad}^{[n-r]}A_2).$$

In particular,

$$(A_1ZA_2)^{[r]} \sqcap B^{[n-r]} = Z^{[r]} \sqcap \big({}^t\mathrm{Ad}^{[n-r]}A_1(B^{[n-r]}){}^t\mathrm{Ad}^{[n-r]}A_2\big).$$

Proof. The assertion (1) is clear by definition. We prove (2). Since both sides of the equality in question are polynomials in the entries of A_1, A_2, it suffices to prove in the case $\det A_1 \neq 0$ and $\det A_2 \neq 0$. Put

$$H = Z^{[r]} \sqcap ({}^t\mathrm{Ad}^{[n-r]}A_1(B^{[s]}\sqcap C^{[t]}){}^t\mathrm{Ad}^{[n-r]}A_2).$$

Then

$$\begin{aligned}&\det A_1 \det A_2 H\\ &= (A_1^{[r]}\sqcap A_1^{[n-r]})\big(Z^{[r]}\sqcap({}^t\mathrm{Ad}^{[n-r]}A_1(B^{[s]}\sqcap C^{[t]}){}^t\mathrm{Ad}^{[n-r]}A_2)\big)(A_2^{[r]}\sqcap A_2^{[n-r]})\\ &= (A_1^{[r]}Z^{[r]}A_2^{[r]})\sqcap\big((A_1^{[n-r]}{}^t\mathrm{Ad}^{[n-r]}A_1)(B^{[s]}\sqcap C^{[t]})({}^t\mathrm{Ad}^{[n-r]}A_2A_2^{[n-r]})\big)\end{aligned}$$

Since we have $A_1^{[n-r]}{}^t\mathrm{Ad}^{[n-r]}A_1 = \det A_1$ and ${}^t\mathrm{Ad}^{[n-r]}A_2A_2^{[n-r]} = \det A_2$, we have

$$\det A_1 \det A_2 H = \det A_1 \det A_2 (A_1^{[r]}Z^{[r]}A_2^{[r]}) \sqcap (B^{[s]}\sqcap C^{[t]}).$$

This proves the assertion. □

From now on for an $n\times n$ matrix A and $\mathbf{i},\mathbf{j}\in\mathcal{I}_{n,r}, \mathbf{i}',\mathbf{j}'\in\mathcal{I}_{n,s}$, we make the convention that

$$A\begin{pmatrix}\mathbf{i}\backslash\mathbf{i}'\\ \mathbf{j}\backslash\mathbf{j}'\end{pmatrix} = 0$$

unless otherwise $\mathrm{supp}(\mathbf{i}') \subset \mathrm{supp}(\mathbf{i})$ and $\mathrm{supp}(\mathbf{j}') \subset \mathrm{supp}(\mathbf{j})$.

Lemma 3.8. *Let $A \in S_n(\mathbb{R})_{>0}$ and $p \geq 1$. Let $\alpha_1 < \cdots < \alpha_r$ and $\beta_1 < \cdots < \beta_r$. Then*

$$({}^t\partial_2 A^{-1}\partial_2)^{[p]}\Big(Z\Big(\begin{smallmatrix}\alpha_1,\dots,\alpha_r\\ \beta_1,\dots,\beta_r\end{smallmatrix}\Big)\Big) = 0.$$

Proof. Since $({}^t\partial_2 A^{-1}\partial_2)^{[p]} = ({}^t\partial_2 A^{-1}\partial_2)^{[p-1]} \sqcap ({}^t\partial_2 a^{-1}\partial_2)^{[1]}$ for $p \geq 1$, it suffices to prove the case $p = 1$. Put $A^{-1} = (a^*_{kl})$. Then the (i,j)-th component of $({}^t\partial_2 A^{-1}\partial_2)^{[1]}$ is

$$\sum_{k,l} a^*_{k,l}\frac{\partial^2}{\partial z_{k,i}\partial z_{l,j}}.$$

The assertion is clear if $r = 1$. Suppose that $r \geq 2$. Since each term of $Z\Big(\begin{smallmatrix}\alpha_1,\dots,\alpha_r\\ \beta_1,\dots,\beta_r\end{smallmatrix}\Big)$ is expressed as a sum of monomials of the form $z_{\alpha_{i_1},\beta_1}\cdots z_{\alpha_{i_r},\beta_r}$ with $i_k \neq i_l$ for $k \neq l$, we have

$$\sum_{k,l} a^*_{k,l}\frac{\partial^2}{\partial z_{k,i}\partial z_{l,i}}\Big(Z\Big(\begin{smallmatrix}\alpha_1,\dots,\alpha_r\\ \beta_1,\dots,\beta_r\end{smallmatrix}\Big)\Big) = 0.$$

Suppose that $i \neq j$. Since $Z\Big(\begin{smallmatrix}\alpha_1,\dots,\alpha_r\\ \beta_1,\dots,\beta_r\end{smallmatrix}\Big)$ is expressed as a sum of polynomials of the form $(z_{ai}z_{bj} - z_{aj}z_{bi})Z\Big(\begin{smallmatrix}(\alpha_1,\dots,\alpha_r)\backslash(a,b)\\ (\beta_1,\dots,\beta_r)\backslash(i,j)\end{smallmatrix}\Big)$, we have

$$a^*_{k,l}\frac{\partial^2}{\partial z_{k,i}\partial z_{l,i}}\Big(Z\Big(\begin{smallmatrix}\alpha_1,\dots,\alpha_r\\ \beta_1,\dots,\beta_r\end{smallmatrix}\Big)\Big) \neq 0$$

if and only if $a = k$ and $b = l$, and hence it is $a^*_{k,l}$. On the other hand for such a, b, $a^*_{l,k}\frac{\partial^2}{\partial z_{l,i}\partial z_{k,i}}\Big(Z\Big(\begin{smallmatrix}\alpha_1,\dots,\alpha_r\\ \beta_1,\dots,\beta_r\end{smallmatrix}\Big)\Big) = -a^*_{l,k} = -a^*_{k,l}$, and hence

$$(a^*_{k,l}\frac{\partial^2}{\partial z_{k,i}\partial z_{l,i}} + a^*_{l,k}\frac{\partial^2}{\partial z_{l,i}\partial z_{k,i}})\Big(Z\Big(\begin{smallmatrix}\alpha_1,\dots,\alpha_r\\ \beta_1,\dots,\beta_r\end{smallmatrix}\Big)\Big) = 0.$$

This proves the assertion. $\square$

Let $\mathcal{J}$ be the ideal of $\mathbb{C}[z_{ij}\ (1 \leq i,j \leq n)]$ generated by $z_{i_1,j_1}z_{i_2,j_2} - z_{i_2,j_1}z_{i_1,j_2}$ $(1 \leq i_1 < i_2 \leq n, 1 \leq j_1 \leq j_2 \leq n)$.

Lemma 3.9. *Let $m = 0$ or a positive integer greater than 1. Then, for any $P \in \mathcal{J}$ we have*

$$(\sum_{i,j} v_i u_j \frac{\partial}{\partial z_{ij}})^m(P)|_{Z=O} = 0.$$

Proof. The assertion is clear if $m = 0$. By Lemma 3.4, it suffices to prove the assertion for $P(Z) = (z_{i1,j_1}z_{i_2,j_2} - z_{i_2,j_1}z_{i_1,j_2})Q(Z)$ with $1 \leq i_1 < i_2 \leq n, 1 \leq j_1 \leq j_2 \leq n$ and $Q(Z)$ a homogeneous polynomial in Z of degree $m-2$. Without loss of generality we may assume $i_1 = j_1 = 1$ and $i_2 = j_2 = 2$. Put $\Delta = z_{11}z_{22} - z_{12}z_{21}$ and

$$\mathcal{P} = \sum_{i,j} v_i u_j \frac{\partial}{\partial z_{ij}}.$$

Then by the Leibniz rule for the differentiation,

$$\mathcal{P}^m(\Delta Q) = \sum_{i=0}^{m} \binom{m}{i} \mathcal{P}^i(\Delta)\mathcal{P}^{m-i}(Q).$$

Since Δ is a homogeneous polynomial of degree 2, we have $\mathcal{P}^i(\Delta) = 0$ for $i \geq 3$ and $\mathcal{P}^i(\Delta)|_{Z=O} = 0$ for $i \leq 1$. Hence

$$\mathcal{P}^m(\Delta Q)|_{Z=O} = 2\mathcal{P}^2(\Delta)|_{Z=O}\mathcal{P}^{m-2}(Q)|_{Z=O}.$$

Put

$$\mathcal{Q} = \mathcal{P}^2 - 2\big(\frac{\partial^2}{\partial z_{11}\partial z_{22}} + \frac{\partial^2}{\partial z_{12}\partial z_{21}}\big)u_1u_2v_1v_2.$$

Then clearly we have

$$\mathcal{Q}(\Delta) = 0,$$

and by a simple computation,

$$\big(\frac{\partial^2}{\partial z_{11}\partial z_{22}} + \frac{\partial^2}{\partial z_{12}\partial z_{21}}\big)(\Delta) = 0.$$

that is, $\mathcal{P}^2(\Delta) = 0$. This completes the proof. $\square$

For $i = 0, \ldots, 2n$ let $\widetilde{P}_{i,\alpha}(T,Z)$ be a homogeneous polynomial in Z such that

$$\widetilde{P}_{i,\alpha}(T,Z) \equiv P_{i,\alpha}(T,Z) \mod \mathcal{J}. \tag{3.7}$$

Let f and T be as in Theorem 3.3. Put

$$\begin{aligned}(3.8)\qquad &\widetilde{Q}_{l-2,\lambda^{(2)},n,n}(T;u,v)\\ &= ((k-2)!)^{-1}\\ &\times \sum_{i=0}^{2n}\sum_{\mu=0}^{(k-l)/2}(-1)^{\mu}\frac{(k-\mu-2)!}{(k-l-2\mu-i)!\mu!}(A[{}^tu]B[{}^tv])^{\mu}\\ &\times (uR{}^tv)^{k-l-2\mu-i}\widetilde{P}_{i,l-2}(T,(u_av_b)_{1\leq a,b\leq n}).\end{aligned}$$

Then, by Lemma 3.9, for $f(W) = \mathbf{e}(\mathrm{tr}(TW))$, we obtain a modification of Theorem 3.3:

$$\begin{aligned}(3.9)\qquad &\widetilde{\mathbb{D}}_{l,\lambda,n,n} \circ \mathcal{D}^2_{n,l-2}(f)\\ &= (2\pi\sqrt{-1})^{k-l+2n}\widetilde{Q}_{l-2,\lambda^{(2)},n,n}(T;u,v)\mathbf{e}(\mathrm{tr}(AZ_1+BZ_4)).\end{aligned}$$

We now seek an explicit formula for $\widetilde{P}_{i,\alpha}(T,Z)$ satisfying (3.7).

Let $\mathbf{a}=(a_1,\ldots,a_r)\in\mathcal{I}_{n,r}$ and $\mathbf{i}=(i_1,\ldots,i_s)\in\mathcal{I}_{n,s}$ with $1\le s\le r\le n$. Suppose that $\mathrm{supp}(\mathbf{i})\subset\mathrm{supp}(\mathbf{a})$. Then we have $i_k=a_{l_k}$ with $1\le l_k\le r$ for any $1\le k\le s$. Then we define $\eta(\mathbf{a};\mathbf{i})$ as

$$\eta(\mathbf{a};\mathbf{i})=(-1)^{l_1+\cdots+l_s}.$$

If $\mathrm{supp}(\mathbf{i})\not\subset\mathrm{supp}(\mathbf{a})$, we put $\eta(\mathbf{a};\mathbf{i})=1$. Let $R=(r_{ij})_{1\le i,j\le n}\in M_n(\mathbb{Z})$ and $A\in\mathcal{H}_n(\mathbb{Z})_{>0}$. Write $A^{-1}=(a^*_{ij})$, and for $\mathbf{a},\mathbf{b}\in\mathcal{I}_{n,p}$ we define $Q^{[1]}\Big(R,A,Z\Big(\begin{smallmatrix}\mathbf{a}\\ \mathbf{b}\end{smallmatrix}\Big)\Big)$ as

$$\begin{aligned}&Q^{[1]}\Big(R,A,Z\Big(\Big(\begin{smallmatrix}\mathbf{a}\\ \mathbf{b}\end{smallmatrix}\Big)\Big)\\&=\sum_{1\le i,j\le n}E_{ij}\Big(\sum_{1\le k,l\le n}a^*_{k,l}\eta(\mathbf{a};k)\eta(\mathbf{b};i)Z\Big(\begin{smallmatrix}\mathbf{a}\setminus k\\ \mathbf{b}\setminus i\end{smallmatrix}\Big)r_{l,j}\\&+\sum_{1\le k,l\le n}a^*_{k,l}\eta(\mathbf{a};l)\eta(\mathbf{b};j)Z\Big(\begin{smallmatrix}\mathbf{a}\setminus l\\ \mathbf{b}\setminus j\end{smallmatrix}\Big)\Big)r_{k,i}\Big)\end{aligned}$$

For $T=\begin{pmatrix}A & R/2\\ {}^tR/2 & B\end{pmatrix}\in S_{2n}(\mathbb{R})$ with $A\in S_n(\mathbb{R})_{>0}$ and $0\le s\le n$, write

$$\mathrm{Ad}^{[s]}(A)(\frac{R}{2})^{[s]}\mathrm{Ad}^{[s]}\big(B-\frac{{}^tR}{2}A^{-1}\frac{R}{2}\big)=(s_{\mathbf{i},\mathbf{j}})_{\mathbf{i},\mathbf{j}\in\mathcal{I}_{n,s}},$$

and for $\mathbf{i},\mathbf{j}\in\mathcal{I}_{n,n-s-l}$ with $l\ge0$ such that $s+l\le n$ put

$$\begin{aligned}S_{n-s,n-s-l}(T;Z)_{\mathbf{i},\mathbf{j}}&=\begin{pmatrix}n\\ n-s\end{pmatrix}^{-1}\frac{(n-s-l)!}{2^{n-s-l}}\\&\times\sum_{\mathbf{a},\mathbf{b}\in\mathcal{I}_{n,n-s}}(-1)^{|\mathbf{a}|+|\mathbf{b}|}\eta(\mathbf{a};\mathbf{i})\eta(\mathbf{b};\mathbf{j})Z\big(\begin{smallmatrix}\mathbf{a}\setminus\mathbf{i}\\ \mathbf{b}\setminus\mathbf{j}\end{smallmatrix}\big)s_{\mathbf{a}^c,\mathbf{b}^c}\end{aligned}$$

and

$$S_{n-s,n-s-l}(T;Z)=\big(S_{n-s,n-s-l}(T;Z)_{\mathbf{i},\mathbf{j}}\big)_{\mathbf{i},\mathbf{j}\in\mathcal{I}_{n,n-s-l}}.$$

The matrix $S_{n-s,n-s-l}(T;Z)$ is an element of $M_{\mathcal{I}_{n,n-s-l}}(\mathbb{Q})$ and its components are homogeneous polynomials of Z of degree l with coefficients in $\mathbb{Q}$. In particular, the components of $S_{n-s,n-s}(T;Z)$ belong to $\mathbb{Q}$, and we sometimes write $S_{n-s}(T)=S_{n-s,n-s}(T;Z)$. Moreover put

$$\begin{aligned}&Q_{n-s,n-s-l}(T;Z)_{\mathbf{i},\mathbf{j}}=\begin{pmatrix}n\\ n-s\end{pmatrix}^{-1}2^{-1}\frac{(n-s-l)!}{2^{n-s-l}}\\&\times\Big(\sum_{\mathbf{a},\mathbf{b}\in\mathcal{I}_{n,n-s}}(-1)^{|\mathbf{a}|+|\mathbf{b}|}\eta(\mathbf{a};\mathbf{i})\eta(\mathbf{b};\mathbf{j})\eta(\mathbf{a}\setminus\mathbf{i};i)\eta(\mathbf{b}\setminus\mathbf{j};j)Z\Big(\begin{smallmatrix}\mathbf{a}\setminus\mathbf{i}\setminus i\\ \mathbf{b}\setminus\mathbf{j}\setminus j\end{smallmatrix}\Big)s_{\mathbf{a}^c,\mathbf{b}^c}\Big)_{1\le i,j\le n}\end{aligned}$$

for $\mathbf{i},\mathbf{j} \in \mathcal{I}_{n,n-s-l}$ with $l \geq 1$. We note that $Q_{n-s,n-s-l}(T;Z)_{\mathbf{i},\mathbf{j}}$ is an $n \times n$ matrix with entries in homogeneous polynomials of degree $l-1$ over $\mathbb{Q}$. We note that

$$\begin{aligned}
&Q_{n-s,n-s-2}(T;Z)_{\mathbf{i},\mathbf{j}} = \binom{n}{n-s}^{-1} 2^{-1} \frac{(n-s-2)!}{2^{n-s-2}} \\
&\times \Bigg(\sum_{(i_1,i_2),(j_1,j_2) \in \mathcal{I}_{n,2}} (-1)^{i_1+i_2+j_1+j_2} \Big(\delta_{i,i_1}\delta_{j,j_1} z_{i_2,j_2} + \delta_{i,i_2}\delta_{j,j_2} z_{i_1,j_1} \\
&- \delta_{i,i_1}\delta_{j,j_2} z_{i_2,j_1} - \delta_{i,i_2}\delta_{j,j_1} z_{i_1,j_2} \Big) s_{(\mathbf{i}\sqcup(i_1,i_2))^c,(\mathbf{j}\sqcup(j_1,j_2))^c} \Bigg)_{1\leq i,j \leq n}
\end{aligned}$$

for $\mathbf{i},\mathbf{j} \in \mathcal{I}_{n,n-s-2}$. Here we make the convention that $s_{(\mathbf{i}\sqcup(i_1,i_2))^c,(\mathbf{j}\sqcup(j_1,j_2))^c} = 0$ if $\mathbf{i} \cap (i_1,i_2) \neq \emptyset$ or $\mathbf{j} \cap (j_1,j_2) \neq \emptyset$. We define $C^{[s]}\left(Z\left(\begin{smallmatrix}\mathbf{a}\\ \mathbf{b}\end{smallmatrix}\right)\right)$ as

$$C^{[s]}\left(Z\left(\begin{smallmatrix}\mathbf{a}\\ \mathbf{b}\end{smallmatrix}\right)\right) = \left(\frac{s!}{2^s}\eta(\mathbf{a};\mathbf{i})\eta(\mathbf{b};\mathbf{j}) Z\left(\begin{smallmatrix}\mathbf{a}\setminus\mathbf{i}\\ \mathbf{b}\setminus\mathbf{j}\end{smallmatrix}\right)\right)_{\mathbf{i},\mathbf{j}\in\mathcal{I}_{n,s}}.$$

Lemma 3.10. (1) *We have*

$$\partial_2^{[s]}\left(Z\left(\begin{smallmatrix}\mathbf{a}\\ \mathbf{b}\end{smallmatrix}\right)\right) = C^{[s]}\left(Z\left(\begin{smallmatrix}\mathbf{a}\\ \mathbf{b}\end{smallmatrix}\right)\right).$$

(2) *Let $A \in S_n(\mathbb{R})_{>0}$ and $g(Z) = \exp(\mathrm{tr}(RZ))$ with $R \in M_n(\mathbb{R})$. Then*

$$\begin{aligned}
&({}^t\partial_2 A^{-1}\partial_2)^{[1]}\left(gZ\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}\right) \\
&= \frac{1}{4} gZ\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix} ({}^tRA^{-1}R)^{[1]} \\
&+ \frac{1}{4} g\ {}^t\left(\eta(\mathbf{a};i)\eta(\mathbf{b};j) Z\begin{pmatrix}\mathbf{a}\setminus i\\ \mathbf{b}\setminus j\end{pmatrix}\right)_{1\leq i,j\leq n} A^{-1}R \\
&+ \frac{1}{4} g\ {}^tRA^{-1}\left(\eta(\mathbf{a};i)\eta(\mathbf{b};j) Z\begin{pmatrix}\mathbf{a}\setminus i\\ \mathbf{b}\setminus j\end{pmatrix}\right)_{1\leq i,j\leq n}.
\end{aligned}$$

Proof. The assertion (1) can be proved by a simple computation. By the Leibniz rule for the differentiation,

$$\begin{aligned}
&({}^t\partial_2 A^{-1}\partial_2)^{[1]}\left(g\ Z\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}\right) \\
&= g({}^t\partial_2 A^{-1}\partial_2)^{[1]}\left(Z\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}\right) + Z\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}({}^t\partial_2 A^{-1}\partial_2)^{[1]}(g) \\
&+ {}^t\partial_2\left(Z\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}\right) A^{-1}\partial_2(g) + {}^t\partial_2(g) A^{-1}\partial_2\left(Z\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}\right).
\end{aligned}$$

We have

$$\partial_2(g)=\frac{1}{2}Rg,\ {}^t\partial_2(g)=\frac{1}{2}{}^tRg,$$

$$({}^t\partial_2A^{-1}\partial_2)^{[1]}(g)=\frac{1}{4}{}^tRA^{-1}Rg,$$

and by Lemma 3.8 we have

$$(\partial_2A^{-1}\partial_2)^{[1]}\Big(Z\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}\Big)=O.$$

Hence, by (1),

$$\begin{aligned}&({}^t\partial_2A^{-1}\partial_2)^{[1]}\Big(g\ Z\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}\Big)\\ &=\frac{1}{4}gZ\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}({}^tRA^{-1}R)^{[1]}\\ &+g\ {}^tC^{[1]}\Big(Z\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}\Big)A^{-1}R+g\ {}^tRA^{-1}C^{[1]}\Big(Z\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}\Big).\end{aligned}$$

This proves the assertion. □

Corollary 3.11. *With the notation of Lemma 3.10, we have*

$$({}^t\partial_2A^{-1}\partial_2)^{[1]}\Big(gZ\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}\Big)=\frac{1}{4}({}^tRA^{-1}R)^{[1]}Z\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}g+\frac{1}{4}Q^{[1]}\Big(Z\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}\Big)g.$$

Proposition 3.12. *Let* $T=\begin{pmatrix}A & R/2\\ {}^tR/2 & B\end{pmatrix}\in S_{2n}(\mathbb{R})$ *with* $A\in S_n(\mathbb{R})_{>0}$.

(1) *Write* $\mathrm{Ad}^{[r]}A(\frac{R}{2})^{[r]}\mathrm{Ad}^{[r]}(B-\frac{1}{4}{}^tRA^{-1}R)=(S(r)_{\mathbf{i},\mathbf{j}})_{\mathbf{i},\mathbf{j}\in\mathcal{I}_{n,r}}$. *Then* $\Delta(r,n-r,T)$ *can be expressed as*

$$\Delta(r,n-r,T)=\begin{pmatrix}n\\ n-r\end{pmatrix}^{-1}\sum_{\mathbf{a},\mathbf{b}\in\mathcal{I}_{n,n-r}}(-1)^{|\mathbf{a}|+|\mathbf{b}|}Z\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix}S(r)_{\mathbf{a}^c,\mathbf{b}^c}.$$

(2) *For* $l\geq 0$, *we have*

$$\partial_2^{[n-s-l]}\big(\Delta(s,n-s,T)\big)=S_{n-s,n-s-l}(T,Z)=\sum_{\mathbf{i},\mathbf{j}\in\mathcal{I}_{n,n-s-l}}S_{n-s,n-s-l}(T;Z)_{\mathbf{i},\mathbf{j}}E_{\mathbf{i},\mathbf{j}},$$

and in particular,

$$\partial_2^{[n-s]}\big(\Delta(s,n-s,T)\big)=S_{n-s}(T)$$

(3) *Let* $g(Z) = \exp(\mathrm{tr}(RZ))$. *Then*

$$\Big(AZ({}^t\partial_2 A^{-1}\partial_2)^{[1]} \sqcap \partial_2^{[n-s-l]}(\Delta(s, n-s, T))\Big)g$$
$$= \frac{1}{2}\sum_{\mathbf{i},\mathbf{j} \in \mathcal{I}_{n,n-s-l}} \Big(\big(AZ({}^tRA^{-1}Q_{n-s,n-s-l}(T;Z) + {}^tQ_{n-s,n-s-l}(T;Z)A^{-1}R)\big)^{[1]} \sqcap E_{\mathbf{i},\mathbf{j}}\Big)g$$
$$+ \frac{1}{4}\big((AZ{}^tRA^{-1}R)^{[1]} \sqcap S_{n-s,n-s-l}(T;Z)\big)g$$

Proof. By Lemma 3.7, (2), we have

$$\Delta(r, n-r, T) = Z^{[n-r]} \sqcap (S(r)_{\mathbf{i},\mathbf{j}})_{\mathbf{i},\mathbf{j} \in \mathcal{I}_{n,r}}.$$

Thus the assertion (1) follows from Lemma 3.7, (1). The assertion (2) follows from Lemma 3.10. Put $c(n,s,l) = \binom{n}{n-s}^{-1}\frac{(n-s-l)!}{2^{n-s-l}}$. Then, by (2), we have

$$(AZ({}^t\partial_2 A^{-1}\partial_2)^{[1]} \sqcap ((\partial_2^{[n-s-l]}(\Delta(s, n-s, T))g)$$
$$= c(n,s,l) \sum_{\mathbf{a},\mathbf{b} \in \mathcal{I}_{n,n-s}} (-1)^{|\mathbf{a}|+|\mathbf{b}|} s_{\mathbf{a}^c,\mathbf{b}^c}$$
$$\times \sum_{\mathbf{i},\mathbf{j} \in \mathcal{I}_{n,n-s-l}} (AZ({}^t\partial_2 A^{-1}\partial_2)^{[1]}\left(\eta(\mathbf{a};\mathbf{i})\eta(\mathbf{b};\mathbf{j})Z\begin{pmatrix}\mathbf{a}\setminus\mathbf{i}\\ \mathbf{b}\setminus\mathbf{j}\end{pmatrix}g\right) \sqcap E_{\mathbf{i},\mathbf{j}}.$$

By Lemma 3.10, (3), we have

$$(AZ({}^t\partial_2 A^{-1}\partial_2)^{[1]}\left(Z\begin{pmatrix}\mathbf{a}\setminus\mathbf{i}\\ \mathbf{b}\setminus\mathbf{j}\end{pmatrix}g\right) = \frac{1}{4}{}^tRA^{-1}RZ\begin{pmatrix}\mathbf{a}\setminus\mathbf{i}\\ \mathbf{b}\setminus\mathbf{j}\end{pmatrix}g$$
$$+ \frac{1}{2}{}^t(\eta(\mathbf{a}\setminus\mathbf{i};i)\eta(\mathbf{b}\setminus\mathbf{j};j)Z\begin{pmatrix}\mathbf{a}\setminus\mathbf{i}\setminus i\\ \mathbf{b}\setminus\mathbf{j}\setminus j\end{pmatrix})_{1\le i,j\le n}A^{-1}R$$
$$+ \frac{1}{2}{}^tA^{-1}(\eta(\mathbf{a}\setminus\mathbf{i};i)\eta(\mathbf{b}\setminus\mathbf{j};j)Z\begin{pmatrix}\mathbf{a}\setminus\mathbf{i}\setminus i\\ \mathbf{b}\setminus\mathbf{j}\setminus j\end{pmatrix})_{1\le i,j\le n}.$$

We note that

$$c(n,s,l)\sum_{\mathbf{i},\mathbf{j} \in \mathcal{I}_{n,n-s-l}} {}^tRA^{-1}R\eta(\mathbf{a};\mathbf{i})\eta(\mathbf{b};\mathbf{j})Z\begin{pmatrix}\mathbf{a}\setminus\mathbf{i}\\ \mathbf{b}\setminus\mathbf{j}\end{pmatrix}g \sqcap E_{\mathbf{i},\mathbf{j}}$$
$$= ({}^tRA^{-1}R)^{[1]} \sqcap S_{n-s,n-s-l}(T,Z)g.$$

This proves the assertion (3). ∎

Let $T = \begin{pmatrix} A & R/2 \\ {}^tR/2 & B \end{pmatrix} \in S_{2n}(\mathbb{R})$ with $A \in S_n(\mathbb{R})_{>0}$. For $i = 0, 1, 2$ define a polynomial $\widetilde{P}_{i,\alpha}(T, Z)$ in Z as follows:

$$\widetilde{P}_{0,\alpha}(T,Z) = (-1)^n C_n(\alpha+1-n+1/2) \sum_{s=0}^{n} (-1)^s \binom{n}{s} C_s(\alpha-n+1/2)$$
$$\times \binom{n}{n-s} S_{n-s,n-s}(T) \sqcap (R/2)^{[s]},$$

$$\widetilde{P}_{1,\alpha}(T,Z) = (-1)^n C_n(\alpha+1-n+1/2) \sum_{s=0}^{n-1} (-1)^s \binom{n}{s} C_s(\alpha-n+1/2)$$
$$\times \binom{n}{n-s-1} S_{n-s,n-s-1}(T;Z) \sqcap (R/2)^{[s+1]}$$
$$+ (-1)^{n-1} n C_{n-1}(\alpha+1-n+1/2) \sum_{s=1}^{n} (-1)^s \binom{n}{s} C_s(\alpha-n+1/2)$$
$$\times \binom{n-1}{n-s} (AZ(B - \frac{1}{4}{}^tRA^{-1}R))^{[1]} \sqcap S_{n-s,n-s}(T) \sqcap (R/2)^{[s-1]}$$
$$+ (-1)^{n-1} n C_{n-1}(\alpha+1-n+1/2) \sum_{s=0}^{n-1} (-1)^s \binom{n}{s} C_s(\alpha-n+1/2) \binom{n-1}{n-s-1}$$
$$\times \Big(\sum_{\mathbf{i},\mathbf{j} \in \mathcal{I}_{n,n-s-1}} (-\frac{1}{2} AZ^tRA^{-1}Q_{n-s,n-s-1}(T)_{\mathbf{i},\mathbf{j}})^{[1]} \sqcap E_{\mathbf{i},\mathbf{j}} \sqcap (R/2)^{[s]}$$
$$+ \sum_{\mathbf{i},\mathbf{j} \in \mathcal{I}_{n,n-s-1}} (-\frac{1}{2} AZ^tQ_{n-s,n-s-1}(T)_{\mathbf{i},\mathbf{j}} A^{-1}R)^{[1]} \sqcap E_{\mathbf{i},\mathbf{j}} \sqcap (R/2)^{[s]} \Big),$$

$$\widetilde{P}_{2,\alpha}(T,Z) = (-1)^{n-1} n C_{n-1}(\alpha+1-n+1/2) \sum_{s=0}^{n-1} (-1)^s \binom{n}{s} C_s(\alpha-n+1/2)$$
$$\times \binom{n-1}{n-s-1} (AZ(B - \frac{1}{4}{}^tRA^{-1}R))^{[1]} \sqcap S_{n-s,n-s-1}(T;Z) \sqcap (R/2)^{[s]}$$
$$+ (-1)^{n-1} n C_{n-1}(\alpha+1-n+1/2) \sum_{s=0}^{n-2} (-1)^s \binom{n}{s} C_s(\alpha-n+1/2) \binom{n-1}{n-s-2}$$
$$\times \Big(\sum_{\mathbf{i},\mathbf{j} \in \mathcal{I}_{n,n-s-2}} (-\frac{1}{2} AZ\,{}^tRA^{-1}Q_{n-s,n-s-2}(T;Z)_{\mathbf{i},\mathbf{j}})^{[1]} \sqcap E_{\mathbf{i},\mathbf{j}} \sqcap (R/2)^{[s+1]}$$
$$+ \sum_{\mathbf{i},\mathbf{j} \in \mathcal{I}_{n,n-s-2}} (-\frac{1}{2} AZ\,{}^tQ_{n-s,n-s-2}(T;Z)_{\mathbf{i},\mathbf{j}} A^{-1}R))^{[1]} \sqcap E_{\mathbf{i},\mathbf{j}} \sqcap (R/2)^{[s+1]} \Big).$$

Then $\widetilde{P}_{i,\alpha}(T,Z)$ is a homogeneous polynomial in Z of degree i.

Theorem 3.13. *Let the notation be as above.*

(1) *Suppose that $i \geq 3$. Then*

$$P_{i,\alpha}(T,Z) \equiv 0 \mod \mathcal{J}.$$

(2) *For $i = 0,1,2$, we have*

$$P_{i,\alpha}(T,Z) \equiv \widetilde{P}_{i,\alpha}(T,Z) \mod \mathcal{J}.$$

Proof. Let $f(W) = \exp(\mathrm{tr}(TW))$. For two matrices $V_1(Z), V_2(Z)$ with entries in polynomials in Z, we write

$$V_1(Z) \equiv V_2(Z) \mod \mathcal{J}$$

if all the components of $V_1(Z) - V_2(Z)$ belong to $\mathcal{J}$. In this case, we also write $V_1(Z)f \equiv V_2(Z)f \mod \mathcal{J}f$. Since

$$\Delta(r,q)(\mathcal{D}_\alpha(f)) \equiv 0 \mod \mathcal{J}f$$

for any $q \geq 2$, we have

$$\begin{aligned}
&\mathcal{D}^2_{n,\alpha}(f)\\
&= \Big((-1)^n C_n(\alpha-n+3/2)\Delta(n,0)\Big(\sum_{s=0}^{n}(-1)^s\binom{n}{s}C_s(\alpha-n+1/2)\Delta(s,n-s,T)f\Big)\\
&+ (-1)^{n-1}C_{n-1}(\alpha-n+3/2)\\
&\times \Delta(n-1,1)\Big(\sum_{s=0}^{n}(-1)^s\binom{n}{s}C_s(\alpha-n+1/2)\Delta(s,n-s,T)f\Big)\Big) \mod \mathcal{J}f.
\end{aligned}$$

We have

$$\begin{aligned}
\Delta(n,0)\big(\Delta(s,n-s,T)f\big) &= \partial_2^{[n]}\big(\Delta(s,n-s,T)f\big)\\
&= \sum_{l=0}^{n}\binom{n}{n-l}\partial_2^{[n-l]}(\Delta(s,n-s,T)) \sqcap \partial_2^{[l]}(f).
\end{aligned}$$

We have

$$\partial_2^{[l]}(f) = (R/2)^{[l]}f.$$

Since $\Delta(s,n-s,T)$ is a homogeneous polynomial of Z of degree $n-s$, we have

$$\partial_2^{[n-l]}(\Delta(s,n-s,T)) = 0 \text{ for } l < s,$$

and by Proposition 3.12, (2)

$$\partial_2^{[n-l]}(\Delta(s,n-s,T)) = S_{n-s,n-l}(T,Z) \text{ for } l \geq s.$$

Since we have $S_{n-s,n-l}(T,Z) \equiv 0 \mod \mathcal{J}$ for $l \geq s+2$, we have

(3.10)

$$\Delta(n,0)\big(\Delta(s,n-s,T)f\big) = \binom{n}{n-s-1} S_{n-s,n-s-1}(T,Z) \sqcap (R/2)^{[s+1]} f + \binom{n}{n-s} S_{n-s,n-s}(T) \sqcap (R/2)^{[s]} f \mod \mathcal{J}f,$$

and the first and the second terms of the right-hand side of (3.10) are homogeneous polynomials in Z of degree 1 and 0, respectively. We have

$$\begin{aligned}
&\Delta(n-1,1)\big(\Delta(s,n-s,T)f\big) \\
&= \Big(\big(AZ(\partial_4 - {}^t\partial_2 A^{-1}\partial_2)\big)^{[1]} \sqcap \partial_2^{[n-1]}\Big)\big(\Delta(s,n-s,T)f\big) \\
&= \Big(\big(AZ\partial_4\big) \sqcap \partial_2^{[n-1]}\Big)\big(\Delta(s,n-s,T)f\big) \\
&- \Big(\big(AZ{}^t\partial_2 A^{-1}\partial_2\big) \sqcap \partial_2^{[n-1]}\Big)\big(\Delta(s,n-s,T)f\big).
\end{aligned}$$

Again, by Proposition 3.12, (2), we have

$$\begin{aligned}
&\partial_2^{[n-1]}(\Delta(s,n-s)f) \\
&= \sum_{l=0}^{n-1} \binom{n-1}{n-l} \partial_2^{[n-l]}(\Delta(s,n-s,T)) \sqcap \partial_2^{[l-1]}(f) \\
&= \sum_{l=0}^{n-1} \binom{n-1}{n-l} S_{n-s,n-l}(T,Z) \sqcap (R/2)^{[l-1]} f \\
&\equiv \binom{n-1}{n-s-1} S_{n-s,n-s-1}(T,Z) \sqcap (R/2)^{[s]} f \\
&+ \binom{n-1}{n-s} S_{n-s,n-s}(T,Z) \sqcap (R/2)^{[s-1]} f \mod \mathcal{J}f.
\end{aligned}$$

Hence we have

(3.11)

$$\begin{aligned}
&\Big(\big(AZ\partial_4\big) \sqcap \partial_2^{[n-1]}\Big)\big(\Delta(s,n-s,T)f\big) \\
&= \binom{n-1}{n-s-1}\Big((AZB)^{[1]} \sqcap S_{n-s,n-s-1}(T,Z) \sqcap (R/2)^{[s]})\Big) f \\
&+ \binom{n-1}{n-s}\Big((AZB)^{[1]} \sqcap S_{n-s,n-s}(T,Z) \sqcap (R/2)^{[s-1]})\Big) f \mod \mathcal{J}f,
\end{aligned}$$

and the first and the second terms of the right-hand side of (3.11) are homogeneous polynomials in Z of degree 2 and 1, respectively. We also have

(3.12)
$$\Big(\big(AZ^t\partial_2 A^{-1}\partial_2\big)\sqcap \partial_2^{[n-1]}\Big)(\Delta(s,n-s,T))f$$
$$\equiv \binom{n-1}{n-s}\Big(\big(AZ^t\partial_2 A^{-1}\partial_2\big)\sqcap \partial_2^{[n-s]}\Big)(\Delta(s,n-s,T))\sqcap (R/2)^{[s-1]}f$$
$$+\binom{n-1}{n-s-1}\Big(\big(AZ^t\partial_2 A^{-1}\partial_2\big)\sqcap \partial_2^{[n-s-1]}\Big)(\Delta(s,n-s,T))\sqcap (R/2)^{[s]}f$$
$$+\binom{n-1}{n-s-2}\Big(\big(AZ^t\partial_2 A^{-1}\partial_2\big)\sqcap \partial_2^{[n-s-2]}\Big)(\Delta(s,n-s,T))\sqcap (R/2)^{[s+1]}f \mod \mathcal{J}f$$

We have

$$\Big(\big(AZ^t\partial_2 A^{-1}\partial_2\big)\sqcap \partial_2^{[n-s]}\Big)(\Delta(s,n-s,T))\sqcap (R/2)^{[s-1]}f$$
$$= \frac{1}{4}(AZ^tRA^{-1}R)^{[1]}\sqcap S_{n-s,n-s}(T)\sqcap (R/2)^{[s-1]}f, \tag{3.13}$$

and it is a homogeneous polynomial in Z of degree 1. Moreover, by Proposition 3.12, (3),

(3.14)
$$\Big(\big(AZ^t\partial_2 A^{-1}\partial_2\big)\sqcap \partial_2^{[n-s-1]}\Big)(\Delta(s,n-s,T))\sqcap (R/2)^{[s]}f$$
$$= \frac{1}{4}(AZ^tRA^{-1}R)^{[1]}\sqcap S_{n-s,n-s-1}(T,Z)\sqcap (R/2)^{[s]}f$$
$$+\frac{1}{2}\sum_{\mathbf{i},\mathbf{j}\in\mathcal{I}_{n,n-s-1}}\big(AZ^tRA^{-1}Q_{n-s,n-s-1}(T)_{\mathbf{i},\mathbf{j}}\big)^{[1]}\sqcap E_{\mathbf{i},\mathbf{j}}\sqcap (R/2)^{[s]}f$$
$$+\frac{1}{2}\sum_{\mathbf{i},\mathbf{j}\in\mathcal{I}_{n,n-s-1}}\big(AZ^tQ_{n-s,n-s-1}(T)_{\mathbf{i},\mathbf{j}}A^{-1}R\big)^{[1]}\sqcap E_{\mathbf{i},\mathbf{j}}\sqcap (R/2)^{[s]}f,$$

and

(3.15)
$$\Big(\big(AZ^t\partial_2 A^{-1}\partial_2\big)\sqcap \partial_2^{[n-s-2]}\Big)(\Delta(s,n-s,T))\sqcap (R/2)^{[s+1]}f$$
$$\equiv \frac{1}{2}\sum_{\mathbf{i},\mathbf{j}\in\mathcal{I}_{n,n-s-2}}\big(AZ^tRA^{-1}Q_{n-s,n-s-2}(T,Z)_{\mathbf{i},\mathbf{j}}\big)^{[1]}\sqcap E_{\mathbf{i},\mathbf{j}}\sqcap (R/2)^{[s+1]}f$$
$$+\frac{1}{2}\sum_{\mathbf{i},\mathbf{j}\in\mathcal{I}_{n,n-s-2}}\big(AZ^tQ_{n-s,n-s-2}(T)_{\mathbf{i},\mathbf{j}}A^{-1}R\big)^{[1]}\sqcap E_{\mathbf{i},\mathbf{j}}\sqcap (R/2)^{[s+1]}f \mod \mathcal{J}f.$$

The first and the second terms of the right-hand side of (3.14) are homogeneous polynomials in Z of degree 2 and 1, respectively, and both terms of the right-hand side of (3.15) are homogeneous polynomials in Z of degree 2. This completes the assertion. □

Corollary 3.14. *Let f and $T = \left(\begin{smallmatrix} A & R/2 \\ {}^tR/2 & B \end{smallmatrix}\right) \in S_{2n}(\mathbb{R})$ be as in Theorem 3.3 with $A \in S_n(\mathbb{R})_{>0}$. Put*

$$\begin{aligned} &\widetilde{Q}_{l-2,\lambda^{(2)},n,n}(T;u,v) = ((k-2)!)^{-1} \\ &\times \sum_{i=0}^{2} \sum_{\mu=0}^{(k-l)/2} (-1)^{\mu} \frac{(k-\mu-2)!}{(k-l-2\mu-i)!\mu!} (A[{}^tu]B[{}^tv])^{\mu} \\ &\times (uR^tv)^{k-l-2\mu-i} \widetilde{P}_{i,l-2}(T,(u_av_b)_{1\le a,b\le n}). \end{aligned}$$

Then

$$\widetilde{\mathbb{D}}_{l,\lambda,n,n} \circ \mathcal{D}^2_{n,l-2}(f) = (2\pi\sqrt{-1})^{k-l+2n} \widetilde{Q}_{l-2,\lambda^{(2)},n,n}(T;u,v)\mathbf{e}(\mathrm{tr}(AZ_1+BZ_4)).$$

Proof. The assertion follows from Theorem 3.13 together with (3.8) and (3.9). □

Corollary 3.15. *Let $W = \begin{pmatrix} Z_1 & Z \\ {}^tZ & Z_4 \end{pmatrix} \in \mathbb{H}_{2n}$ with $Z_1, Z_4 \in \mathbb{H}_n, Z \in M_n(\mathbb{C})$ and $T = \left(\begin{smallmatrix} A & R/2 \\ {}^tR/2 & B \end{smallmatrix}\right) \in S_{2n}(\mathbb{R})$ with $A, B \in S_n(\mathbb{R}), R \in M_n(\mathbb{R})$ and put $f(W) = \mathbf{e}(\mathrm{tr}(TW))$. Let $\widetilde{\mathcal{D}}^2_{n,l-2} = \mathcal{D}^2_{n,l-2}|_{Z=O}$. Then we have*

$$\widetilde{\mathcal{D}}^2_{n,l-2}(f) = \widetilde{P}_{0,l-2}\big(\left(\begin{smallmatrix} A & R/2 \\ {}^tR/2 & B \end{smallmatrix}\right)\big)\mathbf{e}(\mathrm{tr}(AZ_1+BZ_4)).$$

Remark 3.16. An explicit formula for $\widetilde{\mathcal{D}}^2_{n,l-2}$ was already given in [11, Proposition 4.6] in another way.

4. Pullback formula for the Siegel-Eisenstein series

In this section we review the pullback formula for the Siegel-Eisenstein series following [5, 22]. For an even positive integer l, we define the Siegel-Eisenstein series $E_{n,l}(Z,s)$ of degree n as

$$E_{n,l}(Z,s) \;= \textstyle\sum_{M\in \Gamma^{(n)}{}_{\infty}\backslash \Gamma^{(n)}} j(M,Z)^{-l}(\det(\mathrm{Im}(M(Z))))^s$$

$(Z \in \mathbb{H}_n, s \in \mathbb{C})$, where $\Gamma^{(n)}{}_{\infty} = \left\{\begin{pmatrix} * & * \\ 0_n & * \end{pmatrix} \in \Gamma^{(n)}\right\}$. This series converges for $2\,\mathrm{Re}(s)+l > 2n+1$ and is continued meromorphically to the whole plane as a function of s, and belongs to $M_l^{\infty}(\Gamma^{(n)})$. Moreover, it is finite at $s=0$, which will be denoted by $E_{n,l}(Z)$. As for the holomorphy of $E_{n,l}(Z)$ as a function of Z, see [26]. We put

$$E^*_{n,l}(Z) = \zeta(1-l) \prod_{i=1}^{[n/2]} \zeta(1-2l+2i) E_{n,l}(Z).$$

Then put

(4.1)

$$\mathcal{E}_{l-\nu,\lambda^{(\nu)},n,n}(Z_1,Z_2) = \frac{1}{(2\pi\sqrt{-1})^{n\nu+k-l}} \widetilde{\mathbb{D}}_{l-\nu,\lambda^{(\nu)},n,n}(E^*_{2n,l-\nu})(Z_1,Z_2),$$

where $\widetilde{\mathbb{D}}_{l-\nu,\lambda^{(\nu)},n,n}$ is the differential operator defined in (3.6), and $Z_1, Z_2 \in \mathbb{H}_n$.

Now for a Hecke eigenform $F \in S_{\mathbf{k}}(\Gamma^{(n)})$ with $\mathbf{k} = (k, \overbrace{l,\ldots,l}^{n-1})$, let $L(s,F,\mathrm{St})$ be the standard L-function of F, and for an integer r, put

$$\Lambda(r,F,\mathrm{St}) = (-1)^{n(l-r-n)} \prod_{i=0}^{l-n-r-1} C_n(r+i+1/2)$$

$$2^{n(n+2)+2-2nl-r(n+1)-m}(-1)^{r(n+1)/2} \times \frac{\rho_{l,l-r-n}}{(k-l)!} \prod_{j=1}^{n-1} \frac{\Gamma(2l+2j-2n-1)}{\Gamma(2l+j-n-2)}$$
$$\times \frac{\Gamma((k+l)/2-1)\Gamma((k+l)/2-1/2)\Gamma(l-n)\Gamma(k+l-n-1)}{\Gamma(l)\Gamma(l-1/2)\Gamma(l-1)\Gamma(k+l-2)}$$
$$\times \Gamma(r+n) \prod_{j=1}^{n} \Gamma(2r+2n-2j) \frac{L(r,F,\mathrm{St})}{\pi^{nl+k-l+r(n+1)-n(n+1)/2}(F,F)},$$

where

$$\rho_{l,\mu} = \prod_{i=0}^{\mu-1}\prod_{j=1}^{n}(-l+\mu+(j-1)/2-i).$$

(For a precise definition of the standard L-function, see, for example, [2, Section 3].)

Remark 4.1. For a Hecke eigenform F in $S_{\mathbf{k}}(\Gamma^{(n)})$, let $c(F)$ denote the constant defined in [22, Page 263] with the substitutions $k \mapsto l$, $l \mapsto k-l$, and $m \mapsto r$. Then we have

$$\Lambda(r,F,\mathrm{St}) = (-1)^{n(l-r-n)} \prod_{i=0}^{l-n-r-1} C_n(r+i+1/2)(l)_{k-r}$$
$$\times \zeta(1-r-n) \prod_{j=1}^{n} \zeta(1-2r-2n+2j) c(F).$$

Then it is known that $\Lambda(r, F, \mathrm{St})$ belongs to $\mathbb{Q}(F)$ if $\rho(n) \le r \le l-n$ and $r \equiv n \mod 2$, where

$$\rho(n) = \begin{cases} 3 & \text{if } n \equiv 1 \mod 4 \text{ and } n > 1, \\ 1 & \text{otherwise,} \end{cases}$$

and all the Fourier coefficients of F belong to $\mathbb{Q}(F)$ (cf. [4, 24, 22]).

Let $F_1, \ldots, F_d$ be a basis of $S_{\mathbf{k}}(\Gamma^{(n)})$ consisting of Hecke eigenforms, and let $F_{d+1}, \ldots, F_e$ be a basis of the orthogonal complement of $S_{\mathbf{k}}(\Gamma^{(n)})$ in $M_{\mathbf{k}}(\Gamma^{(n)})$ consisting of Hecke eigenforms. Then the following proposition for $k = l$ has been proved in [11, Theorem 4.3] using the pullback formula for the Siegel-Eisenstein series in [6], and in general case it can also be proved in a similar manner using [22] and taking Remarks 3.1, 3.2 and 4.1 into account.

Proposition 4.2. *Let ν be a non-negative integer such that $l - \nu \equiv 0 \mod 2$, and $n + 3 \le l - \nu$ or $n + 1 \le l - \nu$ according as $n \equiv 1 \mod 4$ or not. Then we have*

$$\begin{aligned} &\mathcal{E}_{l-\nu,\lambda^{(\nu)},n,n}(Z_1, Z_2) \\ &= \sum_{i=1}^{d} \Lambda(l-\nu-n, F_i, \mathrm{St})(\overline{F_i(-\overline{Z_1})} \otimes F_i(Z_2)) + \sum_{j=d+1}^{e} \overline{G_j(-\overline{Z_1})} \otimes F_j(Z_2), \end{aligned}$$

where G_j is an element of $M_{\mathbf{k}}(\Gamma^{(n)})$, and in particular, if $\nu > 0$, then $G_j = 0$.

Write $\mathcal{E}_{l-\nu,\lambda^{(\nu)},n,n}(Z_1, Z)$ as

$$\mathcal{E}_{l-\nu,\lambda^{(\nu)},n,n}(Z_1, Z) = \sum_{A \in \mathcal{H}_n(\mathbb{Z})_{\ge 0}} \mathcal{G}_{l-\nu,\lambda^{(\nu)},n,n}(Z, A)\mathbf{e}(\mathrm{tr}(AZ_1)).$$

For an element $P = P(u_1, \ldots, u_n) = \sum_{i_1,\ldots,i_n \ge 0, i_1+\cdots i_n = k-l} c_{i_1,\ldots,i_n} u^{i_1} \cdots u^{i_n} \in V_{n,\lambda}$ with $c_{i_1,\ldots,i_n} \in \mathbb{C}$, put $\bar{P} = \sum_{i_1,\ldots,i_n \ge 0, i_1+\cdots i_n = k-l} \overline{c_{i_1,\ldots,i_n}} u^{i_1} \cdots u^{i_n}$.

Corollary 4.3. *For an element $A \in \mathcal{H}_n(\mathbb{Z})_{>0}$, we have*

$$\begin{aligned} \mathcal{G}_{l-\nu,\lambda^{(\nu)},n,n}(Z, A) = &\sum_{i=1}^{d} \Lambda(l-\nu-n, F_i, \mathrm{St})\overline{a(A, F_i)}F_i(Z) \\ &+ \sum_{j=d+1}^{e} c_{j,A} F_j(Z), \end{aligned}$$

where $c_{j,A}$ is a certain element of $V_{n,\lambda}$. Moreover we have $c_{j,A} = 0$ for any $d + 1 \le j \le e$ if $\nu > 0$.

Proof. This is a consequence of Proposition 4.2. □

For an element $A \in \mathcal{H}_n(\mathbb{Z})_{>0}$, $\mathcal{G}_{l-\nu,\lambda^{(\nu)},n,n}(Z,A)$ can be expressed as

$$\mathcal{G}_{l-\nu,\lambda^{(\nu)},n,n}(Z,A) = \sum_{B \in \mathcal{H}_n(\mathbb{Z})_{\geq 0}} c_{l-\nu,\lambda^{(\nu)},n,n}(A,B)\,\mathbf{e}(\mathrm{tr}(BZ)) \tag{4.2}$$

with $c_{l-\nu,\lambda^{(\nu)},n,n}(A,B) \in V_{n,\lambda} \otimes V_{n,\lambda}$. For our later purpose, we give a formula for $c_{l-\nu,\lambda^{(\nu)},n,n}(A,B)$.

Proposition 4.4. *For any $A, B \in \mathcal{H}_n(\mathbb{Z})_{>0}$, we have*

$$\begin{aligned} &c_{l-\nu,\lambda^{(\nu)},n,n}(A,B) \\ &= \sum_{R \in M_n(\mathbb{Z})} Q_{l-\nu,\lambda^{(\nu)},n,n}\Big(\big(\begin{smallmatrix} A & R/2 \\ {}^tR/2 & B \end{smallmatrix}\big); u, v\Big) a\Big(\big(\begin{smallmatrix} A & R/2 \\ {}^tR/2 & B \end{smallmatrix}\big), E^*_{2n,l-\nu}\Big), \end{aligned}$$

where $Q_{l-\nu,\lambda^{(\nu)},n,n}$ is the polynomial in Theorem 3.3. In particular, if $\nu = 2$, then

$$\begin{aligned} &c_{l-2,\lambda^{(2)},n,n}(A,B) \\ &= \sum_{R \in M_n(\mathbb{Z})} \widetilde{Q}_{l-2,\lambda^{(2)},n,n}\Big(\big(\begin{smallmatrix} A & R/2 \\ {}^tR/2 & B \end{smallmatrix}\big); u, v\Big) a\Big(\big(\begin{smallmatrix} A & R/2 \\ {}^tR/2 & B \end{smallmatrix}\big), E^*_{2n,l-2}\Big), \end{aligned}$$

where $\widetilde{Q}_{l-2,\lambda^{(2)},n,n}$ is the polynomial in (3.8).

The following lemma will be used to prove Theorem 5.5.

Lemma 4.5. *For elements $A, B \in \mathcal{H}_n(\mathbb{Z})_{>0}$, $c_{l-\nu,\lambda^{(\nu)},n,n}(A,B)$ belongs to $(V_{n,\lambda} \otimes V_{n,\lambda})(K)$, and in particular, if p is a prime number such that $p > k + l$, then it belongs to $(V_{n,\lambda} \otimes V_{n,\lambda})(\mathbb{Z}_{(p)})$.*

Proof. The assertion follows from Proposition 4.4 and [2, Proposition 6.5]. □

5. Applications

In this section we give exact standard L-values for some vector-valued Hecke eigenforms of degree 3 and give congruences between them; see Theorems 5.3 and 5.5. Supplementary Mathematica notebooks are available at [25] (see the directory `applications/Ikeda-Katsurada-Lee-2025-differential-operator`). These notebooks implement Proposition 4.4 together with (3.8) to compute the coefficients $c_{l-\nu,\lambda^{(\nu)},n,n}(\cdot,\cdot)$ in the case $n = 3$, $\nu = 2$, and $\lambda = (k - l, 0, 0)$. The Fourier coefficients of the Siegel–Eisenstein series are computed in these notebooks using an explicit formula for the Siegel series [7, 17].

Let q be a prime number and $F \in M_{(k,l,l)}(\Gamma^{(3)})$. In general, $a(A, F|T(q))(u_1, u_2, u_3)$ is given by

$$\begin{aligned} a(A, F|T(q))(u_1, u_2, u_3) &= a(qA, F)(u_1, u_2, u_3) \\ &+ q^{l-3} \sum_{D \in \mathrm{GL}_3(\mathbb{Z})(q1_2 \perp 1)\mathrm{GL}_3(\mathbb{Z})/\mathrm{GL}_3(\mathbb{Z})} a(q^{-1}A[D], F)(q(u_1, u_2, u_3)\ {}^tD^{-1}) \\ &+ q^{2l-5} \sum_{D \in \mathrm{GL}_3(\mathbb{Z})(q \perp 1_2)\mathrm{GL}_3(\mathbb{Z})/\mathrm{GL}_3(\mathbb{Z})} a(q^{-1}A[D], F)(q(u_1, u_2, u_3)\ {}^tD^{-1}) \\ &+ q^{3l-6} a(q^{-1}A, F)(u_1, u_2, u_3). \end{aligned}$$

Let $A_1 = \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/2 \\ 0 & 1/2 & 1 \end{pmatrix}$. For $A \in \{A_1, A_2\}$, we have

$$\begin{aligned} a(A, F|T(q))(u_1, u_2, u_3) &= a(qA, F)(u_1, u_2, u_3) \\ &+ q^{l-3} \sum_{D \in \mathrm{GL}_3(\mathbb{Z})(q1_2 \perp 1)\mathrm{GL}_3(\mathbb{Z})/\mathrm{GL}_3(\mathbb{Z})} a(q^{-1}A[D], F)(q(u_1, u_2, u_3)\ {}^tD^{-1}). \end{aligned}$$

To write $a(A, F|T(q))(u_1, u_2, u_3)$ explicitly for $q = 2, 3$ and for the matrices A_1, A_2, we fix the following coset representatives:

$$L_0 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}.$$

For $a, b \in \mathbb{Z}$ we also define

$$L_{a,b} = \begin{pmatrix} 1 & 0 & 0 \\ a & 3 & 0 \\ b & 0 & 3 \end{pmatrix}.$$

With these representatives, we can write $a(A, F|T(q))(u_1, u_2, u_3)$ explicitly for $q = 2, 3$ and $A \in \{A_1, A_2\}$ as follows:
(5.1)

$$\begin{aligned}
a(A_1, F|T(2))(u_1, u_2, u_3) &= a(2A_1, F)(u_1, u_2, u_3) \\
&\quad + 2^{l-3} a(2^{-1}A_1[L_0], F)\big(2(u_1, u_2, u_3)\, {}^t L_0^{-1}\big), \\
a(A_2, F|T(2))(u_1, u_2, u_3) &= a(2A_2, F)(u_1, u_2, u_3) \\
&\quad + 2^{l-3} \sum_{i=0}^{2} a(2^{-1}A_2[L_i], F)\big(2(u_1, u_2, u_3)\, {}^t L_i^{-1}\big), \\
a(A_1, F|T(3))(u_1, u_2, u_3) &= a(3A_1, F)(u_1, u_2, u_3) \\
&\quad + 3^{l-3} \sum_{(a,b)=(-1,1),(0,1),(-1,-1)} a(3^{-1}A_2[L_{a,b}], F)\big(3(u_1, u_2, u_3)\, {}^t L_{a,b}^{-1}\big) \\
&\quad + 3^{l-3} a(3^{-1}A_1[M_1], F)\big(3(u_1, u_2, u_3)\, {}^t M_1^{-1}\big), \\
a(A_2, F|T(3))(u_1, u_2, u_3) &= a(3A_2, F)(u_1, u_2, u_3) \\
&\quad + 3^{l-3} a(3^{-1}A_2[M_1], F)\big(3(u_1, u_2, u_3)\, {}^t M_1^{-1}\big).
\end{aligned}$$

Let $\mathbf{k} = (k, l, l)$ and $\lambda = (k-l, 0, 0)$. Let $V_{3,\lambda}^{(1)}$ (resp. $V_{3,\lambda}^{(2)}$) be the space of homogeneous polynomials in u_1, u_2, u_3 (resp. in v_1, v_2, v_3) of degree $k - l$. Let

$$F = F(Z, u_1, u_2, u_3) = \sum_A a(A, F)(u_1, u_2, u_3)\mathbf{e}(\mathrm{tr}(AZ))$$

be an element of $S_{\mathbf{k}}(\Gamma^{(3)})$. Then, F can be regarded as a mapping from $\mathbb{H}_3$ to $V_{3,\lambda}^{(1)}$.

Let us consider the case $(k, l, l) = (16, 10, 10), (16, 12, 12), (20, 8, 8)$ (cf. [3, Table 3]). In all these cases, $\dim S_{(k,l,l)}(\Gamma^{(3)}) = 2$ and is spanned by two Hecke eigenforms F_1 and F_2. To construct them, we first obtain two auxiliary forms in $S_{\mathbf{k}}(\Gamma^{(3)})$ by the pullback method, which are not necessarily Hecke eigenforms, and then write F_1 and F_2 in terms of these forms.

To describe their construction, let $A \in \mathcal{H}_3(\mathbb{Z})_{>0}$ and, as in Section 4, define

$$\mathcal{G}_{l-2,\lambda^{(2)},3,3}(Z, A)(u_1, u_2, u_3, v_1, v_2, v_3) \in V_{3,\lambda}^{(1)} \otimes S_{\mathbf{k}}(\Gamma^{(3)}).$$

For any $(b_1, b_2, b_3) \in \mathbb{C}^3$, its specialization at $(u_1, u_2, u_3) = (b_1, b_2, b_3)$ lies in $S_{\mathbf{k}}(\Gamma^{(3)})$. We shall henceforth write

$$G_A(Z)(v_1, v_2, v_3) := \mathcal{G}_{l-2,\lambda^{(2)},3,3}(Z, A)(1, 1, 1, v_1, v_2, v_3).$$

Now consider the two specializations G_{A_1} and G_{A_2}.

Proposition 5.1. *The two forms G_{A_1} and G_{A_2} are linearly independent, and form a basis of $S_{\mathbf{k}}(\Gamma^{(3)})$.*

Proof. Note that the B-th Fourier coefficient of G_A is given by

$$c_{l-2,\lambda^{(2)},3,3}(A,B)(1,1,1,v_1,v_2,v_3)$$

as in (4.2).

For $A, B \in \{A_1, A_2\}$ set

$$C_G = \big(c_{l-2,\lambda^{(2)},3,3}(A_i,A_j)(1,1,1,1,1,1)\big)_{1\le i,j\le 2}$$

and compute it explicitly with the aid of Mathematica (cf. [25]) using Proposition 4.4. The results are given in Table 1 below. In particular,

Table 1. The 2×2 matrix C_G for three weights $\mathbf{k}$.

$\mathbf{k}$	C_G
$(16,10,10)$	$\begin{pmatrix} \frac{128550275}{16} & \frac{37738575}{2} \\ \frac{37738575}{2} & 90794844 \end{pmatrix}$
$(16,12,12)$	$\begin{pmatrix} \frac{120297398571}{8} & 6004811043 \\ 6004811043 & 188635073022 \end{pmatrix}$
$(20,8,8)$	$\begin{pmatrix} \frac{53922613}{8568} & \frac{4998311}{2856} \\ \frac{4998311}{2856} & \frac{25638581}{7616} \end{pmatrix}$

$\det C_G \neq 0$ in each case (see also Table 1), hence G_{A_1} and G_{A_2} are linearly independent. As the space $S_{\mathbf{k}}(\Gamma^{(3)})$ is two-dimensional in these cases, they form a basis. □

Next, for a prime q, we compute the matrix $M_{T(q)}$ of the Hecke operator $T(q)$ with respect to the ordered basis $\{G_{A_1}, G_{A_2}\}$ of $S_{\mathbf{k}}(\Gamma^{(3)})$. Using (5.1), we first compute the Fourier coefficients of $G_{A_1}|T(q)$ and $G_{A_2}|T(q)$ at A_1 and A_2 and obtain the matrix

$$C_{G|T(q)} = \big(a(A_i, G_{A_j}|T(q))(1,1,1)\big)_{1\le i,j\le 2}$$

given in Table 2 below. The following relation holds:

$$C_G\, M_{T(q)} = C_{G|T(q)}.$$

Consequently, the matrices $M_{T(q)}$ are given by Table 3 below.

By diagonalizing the above matrices $M_{T(p)}$, we can write down $T(p)$-eigenforms F_1 and F_2 in $S_{\mathbf{k}}(\Gamma^{(3)})$ explicitly as linear combinations of G_{A_1} and G_{A_2}. The coefficients c_i, d_i in $F_1 = c_1 G_{A_1} + c_2 G_{A_2}$ and $F_2 = d_1 G_{A_1} + d_2 G_{A_2}$ are given in Table 4. Once we have the explicit expressions of F_1 and F_2, we can compute their Fourier coefficients at A_1 and A_2 by using the matrix C_G above. Since $\lambda_{F_1}(T(q)) \neq \lambda_{F_2}(T(q))$ in all three cases, and all Hecke operators commute with $T(q)$, each Hecke

$\mathbf{k}$, q	$C_{G\|T(q)}$
$(16,10,10)$, $q=2$	$\begin{pmatrix} 273143581200 & 1779907893120 \\ 1779907893120 & 11190327386112 \end{pmatrix}$
$(16,12,12)$, $q=2$	$\begin{pmatrix} 6431460509205144 & 4152056017386816 \\ 4152056017386816 & 17465090828955264 \end{pmatrix}$
$(20,8,8)$, $q=3$	$\begin{pmatrix} 345670315668 & 13651687188 \\ 13651687188 & \frac{13194153261}{2} \end{pmatrix}$

TABLE 2. The 2×2 matrix $C_{G|T(q)}$ for three weights $\mathbf{k}$.

$\mathbf{k}$, q	$M_{T(q)}$ on $\{G_{A_1}, G_{A_2}\}$
$(16,10,10)$, $q=2$	$\begin{pmatrix} -\frac{6516864}{277} & -\frac{183762432}{1385} \\ \frac{6784560}{277} & \frac{41777856}{277} \end{pmatrix}$
$(16,12,12)$, $q=2$	$\begin{pmatrix} \frac{47946816}{113} & \frac{27371520}{113} \\ \frac{960960}{113} & \frac{9590976}{113} \end{pmatrix}$
$(20,8,8)$, $q=3$	$\begin{pmatrix} \frac{5219796384}{83} & \frac{157593600}{83} \\ -\frac{45163929600}{1577} & \frac{80724384}{83} \end{pmatrix}$

TABLE 3. The 2×2 matrix $M_{T(q)}$ for three weights $\mathbf{k}$.

operator preserves the two 1-dimensional $T(q)$-eigenspaces. Hence F_1 and F_2 are Hecke eigenforms, not just $T(q)$-eigenforms. Thus, with G_{A_1} and G_{A_2} in hand, we now construct a Hecke eigenbasis F_1, F_2.

Remark 5.2. In case $\mathbf{k} = (20,8,8)$, we have $\lambda_{F_1}(T(2)) = \lambda_{F_2}(T(2))$. Since $T(2)$ does not distinguish F_1 and F_2, we work with $T(3)$ instead.

$\mathbf{k}$	(16,10,10)	(16,12,12)	(20,8,8)
$u = (u_1, u_2, u_3)$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$
(c_1, c_2)	$(-\frac{2}{203595}, \frac{349}{30783564})$	$(\frac{16}{3155328945}, \frac{13}{104125855185})$	$(-\frac{91392}{272987}, \frac{814912}{5186753})$
(d_1, d_2)	$(-\frac{3376}{9704695}, \frac{108}{1940939})$	$(-\frac{8}{3984467123}, \frac{44}{15368658903})$	$(-\frac{8}{3735}, \frac{14656}{212895})$
$a(A_1, F_1)(u)$	135	77	-1832
$a(A_2, F_1)(u)$	844	54	-57
$a(A_1, F_2)(u)$	-1745	-13	107
$a(A_2, F_2)(u)$	-1512	528	228
q	2	2	3
$\lambda_{F_1}(T(q))$	129600	430272	61998048
$\lambda_{F_2}(T(q))$	-2304	78912	1863648
$a(A_1, G_{A_1})(u)$	$\frac{128550275}{16}$	$\frac{120297398571}{8}$	$\frac{53922613}{8568}$
$a(A_1, G_{A_1}\|T(q))(u)$	273143581200	6431460509205144	345670315668

TABLE 4. Summary of explicit numerical data.

Now we consider the standard L-values $\Lambda(s, F_i, \mathrm{St})$ for $i = 1, 2$.

Theorem 5.3. *Let F_1 and F_2 be as above. Then*

$$\Lambda(l-5, F_1, \mathrm{St}) = \begin{cases} \frac{3^4 \cdot 7^3}{229} & \mathbf{k} = (16, 10, 10) \\ \frac{3^3 \cdot 5 \cdot 11 \cdot 17 \cdot 23^3}{2 \cdot 61} & \mathbf{k} = (16, 12, 12) \\ \frac{11 \cdot 13 \cdot 23}{2^6 \cdot 3^2 \cdot 7 \cdot 17 \cdot 29} & \mathbf{k} = (20, 8, 8), \end{cases}$$

and

$$\Lambda(l-5, F_2, \mathrm{St}) = \begin{cases} \frac{7^2 \cdot 11 \cdot 13}{2^4 \cdot 229} & \mathbf{k} = (16, 10, 10) \\ \frac{3^2 \cdot 7 \cdot 13 \cdot 17 \cdot 23 \cdot 991}{2^3 \cdot 61} & \mathbf{k} = (16, 12, 12) \\ \frac{3 \cdot 5}{2^3 \cdot 29} & \mathbf{k} = (20, 8, 8). \end{cases}$$

Proof. Corollary 4.3 implies that
(5.2)
$$c_{l-2,\lambda^{(2)},3,3}(A, B)(u, v) = \sum_{i=1}^{2} \Lambda(l-5, F_i, \mathrm{St}) \, \overline{a(A, F_i)(u)} \, a(B, F_i)(v),$$

Specializing to $A, B \in \{A_1, A_2\}$ and evaluating at $(u_1, u_2, u_3) = (1, 1, 1)$ and $(v_1, v_2, v_3) = (1, 1, 1)$, we obtain an over-determined system of linear equations for the two unknowns $\Lambda(l-5, F_1, \mathrm{St})$ and $\Lambda(l-5, F_2, \mathrm{St})$. Using the Fourier coefficients listed in Table 4, we find that the system is consistent, and the standard L-values can be obtained by solving it. □

Let F and G be Hecke eigenforms in $M_{\mathbf{k}}(\Gamma^{(n)})$. Let K be an algebraic number field of finite degree containing $\mathbb{Q}(F) \cdot \mathbb{Q}(G)$, $\mathfrak{o}$ its ring of integers, and $\mathfrak{p}$ a prime ideal of $\mathfrak{o}$. Then we write

$$G \equiv_{\mathrm{ev}} F \mod \mathfrak{p}$$

if

$$\lambda_G(T) \equiv \lambda_F(T) \mod \mathfrak{p} \text{ for any } T \in \mathbf{L}_n^{(\mathbf{k})}.$$

Lemma 5.4. *Let H_1, H_2 be Hecke eigenforms in $M_{\mathbf{k}}(\Gamma^{(n)})$. Let K be an algebraic number field of finite degree containing $\mathbb{Q}(H_1) \cdot \mathbb{Q}(H_2)$, $\mathfrak{o}$ its ring of integers, and $\mathfrak{p}$ a prime ideal of $\mathfrak{o}$. Let*

$$G = G(Z) = \sum_{A} a(A, G) \mathbf{e}(\mathrm{tr}(AZ)) \quad (a(A, G) \in V_{n,\lambda})$$

be an element of $M_{\mathbf{k}}(\Gamma^{(n)})(\mathfrak{o}_{(\mathfrak{p})})$. Suppose that

(1) *G can be expressed as*

$$G = c_1 H_1 + c_2 H_2 \text{ with } c_1, c_2 \in K.$$

(2) *For some Hecke operator* T,

$$\lambda_{H_1}(T) \equiv \lambda_{H_2}(T) \mod \mathfrak{p}$$

and

$$a(A,G)\lambda_{H_1}(T) - a(A,G|T) \not\equiv 0 \mod \mathfrak{p}$$

for some $A \in \mathcal{H}_n(\mathbb{Z})_{>0}$.

Then

$$H_1 \equiv_{\mathrm{ev}} H_2 \mod \mathfrak{p}.$$

Proof. By the assumption, $\mathrm{ord}_{\mathfrak{p}}(c_1) < 0$ and $\mathrm{ord}_{\mathfrak{p}}(c_2) < 0$. Then, the assertion can be proved using the same argument as in the proof of [18, Lemma 5.1]. (See also [2, Lemma 6.10].) □

Now we have the following theorem.

Theorem 5.5. *For* (k,l,l) *above, put*

$$\mathfrak{p} = \begin{cases} (229) & \mathbf{k} = (16,10,10), \\ (61) & \mathbf{k} = (16,12,12), \\ (29) & \mathbf{k} = (20,8,8). \end{cases}$$

Then

$$F_1 \equiv_{\mathrm{ev}} F_2 \mod \mathfrak{p}.$$

Proof. Put $G = G_{A_1}$. The concrete numerical values used here are summarized in Table 4. Then we have

$$G = c_1F_1 + c_2F_2 \text{ with } c_1, c_2 \in \mathbb{Q}.$$

In all three cases, we can check that $\lambda_{F_2}(T(q)) - \lambda_{F_1}(T(q))$ is congruent to 0 modulo $\mathfrak{p}$. And $a(A_1,G)\lambda_{F_1}(T(q)) - a(A_1,G|T(q))$ is not congruent to zero modulo $\mathfrak{p}$. Thus by Lemma 5.4, we have

$$F_1 \equiv_{\mathrm{ev}} F_2 \mod \mathfrak{p}.$$

□

Remark 5.6. (1) In all the three cases above one of F_1 and F_2 is the (generalized) Ikeda-Miyawaki lift (cf. [23], [15], [2, Theorem 4.3]) and the other is a stable tempered form. This type of congruence was treated in the scalar-valued case by [13]. In a subsequent paper [16], we will investigate this type of congruence in more general setting.

(2) Theorem 5.3 may be also proved using $\widetilde{\mathbb{D}}_{l,\lambda,3,3}$. But in that case, in view of Corollary 4.3, we have to also consider congruence or non-congruence between cuspidal and non-cuspidal Hecke eigenforms. The use of $\widetilde{\mathbb{D}}_{l-2,\lambda^{(2)},3,3}$ can avoid such an annoying task.

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