Kirillov-Reshetikhin modules and fusion rings from CFT

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Q-system : definition

Definition

Let X be a Dynkin diagram of type ADE and I be the set of its vertices. For a family of variables $\{Q_m^{(a)} | a \in I, m \in \mathbb{Z}_{\geq 0}\}$ in a commutative ring (like $\mathbb{C}$), consider recurrences given by

$$\left(Q_m^{(a)}\right)^2 = \prod_{b \in I, b \sim a} Q_m^{(b)} + Q_{m-1}^{(a)} Q_{m+1}^{(a)}$$

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We call this system the unrestricted Q-system of type X. We use boundary conditions $Q_0^{(a)} = 1$ for all $a \in I$.

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- There is a more general and complicated definition of the Q-system associated to a multiply-laced Dynkin diagram

Q-system : A_4 example

Dynkin diagram : $\bullet_{(1)} - \bullet_{(2)} - \bullet_{(3)} - \bullet_{(4)}$
Use the recursion

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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

- Question : Fix a positive integer $k \geq 1$ (level). How can we find $(Q^{(a)}_1)_{a\in I}$ such that $Q^{(a)}_k = 1$ for all $a \in I$?

level k restricted Q-system

Definition

For variables $\left(Q_m^{(a)}\right)$ with $0 \le m \le k$ and $a \in I$, consider a system of equations

$$\begin{cases} Q_0^{(a)} = 1 & a \in I \\ \left(Q_m^{(a)}\right)^2 = \prod_{b \sim a} \left(Q_m^{(b)}\right) + Q_{m-1}^{(a)} Q_{m+1}^{(a)} & 1 \le m \le k-1, a \in I \\ Q_k^{(a)} = 1 & a \in I \end{cases}$$

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- It is known that there exists a unique positive solution of the level k restricted Q-system over $\mathbb{C}$.
- Sometimes, we replace the last condition with $Q_{k+1}^{(a)} = 0$ for each $a \in I$.

Rogers dilogarithm function

The Rogers dilogarithm function is defined by

$$L(x) = -\frac{1}{2}\int_0^x \frac{\log(1-y)}{y} + \frac{\log(y)}{1-y} dy$$

for $x \in (0,1)$. We set $L(0) = 0$ and $L(1) = \pi^2/6$ so that L is continuous on $[0,1]$.

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- many functional identities are satisfied. For example,

$$L(x)+L(1-xy)+L(y)+L(\frac{1-y}{1-xy})+L\left(\frac{1-x}{1-xy}\right) = \frac{\pi^2}{2} = 3L(1)$$

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- shows up in many places (e.g. number theory, algebraic K-theory, hyperbolic geometry)
- computes central charges for some conformal field theories

an equation from TBA and Y-system

For variables $\{f_m^{(a)} | a \in I, 1 \leq m \leq k-1\}$, consider a system of equations given by

$$\sum_{b \in I} \mathcal{C}(X)_{ab} \log(1 - f_m^{(b)}) = \sum_{n=1}^{k-1} \mathcal{C}(A_{k-1})_{mn} \log f_n^{(a)}.$$

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Note that $f_m^{(a)} = \frac{Y_m^{(a)}(\infty)}{1+Y_m^{(a)}(\infty)}$ where $\left(Y_m^{(a)}(\infty)\right)$ is a u-independent solution of the Y-system

$$Y_m^{(a)}(u-1) Y_m^{(a)}(u+1) = \frac{\prod_{b \in I : b \sim a}(1 + Y_m^{(b)}(u))}{(1 + Y_{m-1}^{(a)}(u)^{-1})(1 + Y_{m+1}^{(a)}(u)^{-1})}, \quad u \in \mathbb{Z}$$

with the boundary conditions

$$(Y_0^{(a)}(u))^{-1} = (Y_k^{(a)}(u))^{-1} = 0$$

TBA and dilogarithm identities for conformal field theories

Theorem (Bazhanov,Kirillov,Reshetikhin '87, $\cdots$, Nakanishi '10)

Let X be a Dynkin diagram of type ADE of rank r and $\mathfrak{g}$ be the corresponding simple Lie algebra. Let $(f_m^{(a)})$ be the unique positive solution of the above such that $0 < f_m^{(a)} < 1$. Then

$$\frac{6}{\pi^2}\sum_{a\in I}\sum_{m=1}^{k-1} L(f_m^{(a)}) = \frac{k\dim\mathfrak{g}}{h+k} - r = \frac{(k-1)hr}{h+k}$$

where h denotes the Coxeter number of X.

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- Originated from the Thermodynamic Bethe ansatz (TBA) calculations for some models in statistical mechanics and had been open for many years

TBA and level restricted Q-system

The unique positive solution of the equations

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such that $0 < f_m^{(a)} < 1$ can be constructed from the unique positive solution of level k restricted Q-system :

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such that $0 < f_m^{(a)} < 1$ can be constructed from the unique positive solution of level k restricted Q-system :

$$f_m^{(a)} = 1 - \frac{Q_{m-1}^{(a)} Q_{m+1}^{(a)}}{(Q_m^{(a)})^2} = \frac{\prod_{b\sim a} Q_m^{(b)}}{(Q_m^{(a)})^2}.$$

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- The quantized universal enveloping algebra $U_q(\mathfrak{g})$ is contained in $U_q(\hat{\mathfrak{g}})$ as a subalgebra
- lifting of an irreducible $U_q(\mathfrak{g})$-module of highest weight $m\omega_a$ to an $U_q(\hat{\mathfrak{g}})$-module by adding more irreducible $U_q(\mathfrak{g})$-modules (minimal affinization)

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- Let $Q_m^{(a)}$ be the classical character of $\mathrm{res}\, W_m^{(a)}(u)$ and then $Q_m^{(a)}$ can be written as $\sum_{\lambda \in P_+} Z(a, m, \lambda)\chi_\lambda$ for some $Z(a, m, \lambda) \in \mathbb{Z}_{\geq 0}$ where χ_λ is the character of a highest weight representation

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Theorem (Nakajima '03, Hernandez '06)

The classical characters $\{Q_m^{(a)} | a \in I, m \in \mathbb{Z}_{\geq 0}\}$ satisfy the unrestricted Q-system :

$$\left(Q_m^{(a)}\right)^2 = \prod_{b \sim a} Q_m^{(a)} + Q_{m-1}^{(a)} Q_{m+1}^{(a)}.$$

character solutions of the unrestricted Q-systems

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For $X = D_r$,

$$Q^{(a)}_m = \begin{cases} \sum \chi_{k_a\omega_a + k_{a-2}\omega_{a-2} + \cdots + k_1\omega_1} & 1 \leq a < r-1, a \equiv 1 \pmod 2, \\ \sum \chi_{k_a\omega_a + k_{a-2}\omega_{a-2} + \cdots + k_0\omega_0} & 1 \leq a < r-1, a \equiv 0 \pmod 2, \\ \chi_{m\omega_a} & a = r-1, r \end{cases}$$

where $\omega_0 = 0$ and the summation is over all nonnegative integers satisfying $k_a + k_{a-2} + \cdots + k_1 = m$ for a odd and $k_a + k_{a-2} + \cdots + k_0 = m$ for a even.

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All known for classical types and partially known for exceptional types

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- For a given level $k \geq 1$, which elements can satisfy $Q_k^{(a)} = 1$ for all $a \in I$?
- Among those elements which one satisfies $Q_m^{(a)} > 0$ for all $0 \leq m \leq k$ and $a \in I$?

Kuniba-Nakanishi-Suzuki (KNS) conjecture

Let $\rho = \sum_{i=1}^{r} \omega_i \in P$ the Weyl vector.

Conjecture (Kirillov '89, Kuniba-Nakanishi-Suzuki '92)

Let $\mathcal{D}_m^{(a)} = Q_m^{(a)}(\frac{\rho}{h+k})$ for each (a, m). It satisfies the following properties :

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It has been known to be true for $X = A_r$.

numerical example : $X = D_5, k = 4$

$$
\begin{bmatrix}
1.000 & 1.000 & 1.000 & 1.000 & 1.000 \\
3.732 & 8.464 & 14.93 & 4.732 & 4.732 \\
5.464 & 15.93 & 33.32 & 7.464 & 7.464 \\
3.732 & 8.464 & 14.93 & 4.732 & 4.732 \\
1.000 & 1.000 & 1.000 & 1.000 & 1.000 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1.000 & 1.000 & 1.000 & 1.000 & 1.000 \\
3.732 & 8.464 & 14.93 & 4.732 & 4.732 \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}
$$

from finite to affine by adding an extra vertex

For a given classical weight $\lambda = \sum_{i=1}^{r} \lambda_i \omega_i \in P$, define its level k affinization $\hat{\lambda} = \sum_{i=0}^{r} \lambda_i \hat{\omega}_i \in \hat{P}_k$ where

- $\hat{P} = \mathbb{Z}\hat{\omega}_0 \oplus \cdots \oplus \mathbb{Z}\hat{\omega}_r$ the affine weight lattice

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from finite to affine by adding an extra vertex

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Example ($X = D_5, k = 4$)

$$\mathcal{D}_4^{(2)} = \chi_0 + \chi_{\omega_2} + \chi_{2\omega_2} + \chi_{3\omega_2} + \chi_{4\omega_2} \text{ evaluated at } \frac{\rho}{h+k}$$

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$$\begin{aligned}\mathcal{D}_4^{(2)} &= \chi_0 + \chi_{\omega_2} + \chi_{2\omega_2} + \chi_{3\omega_2} + \chi_{4\omega_2} \text{ evaluated at } \frac{\rho}{h+k} \\ &= \mathcal{D}_{4\hat{\omega}_0} + \mathcal{D}_{2\hat{\omega}_0+\hat{\omega}_2} + \mathcal{D}_{2\hat{\omega}_2} + \mathcal{D}_{-2\hat{\omega}_0+3\hat{\omega}_2} + \mathcal{D}_{-4\hat{\omega}_0+4\hat{\omega}_2} \\ &= 1?\end{aligned}$$

For $\hat{\lambda} \in \hat{P}_k$, $\mathcal{D}_{\hat{\lambda}}$ is not positive in general!

quantum dimensions

Definition

For $\hat{\lambda} \in \hat{P}_k$, the quantum dimension or q-dimension of $\hat{\lambda}$ is defined by

$$\mathcal{D}_{\hat{\lambda}} = \chi_\lambda \left(\frac{\rho}{h+k} \right) = \frac{\prod_{\alpha>0} \sin \frac{\pi(\lambda+\rho|\alpha)}{h+k}}{\prod_{\alpha>0} \sin \frac{\pi(\rho|\alpha)}{h+k}}$$

where $(\cdot|\cdot)$ is the standard bilinear form on P such that $(\theta|\theta) = 2$.

affine Weyl group

- The affine Weyl group $\hat{W}$ is generated by $s_0, s_1, \cdots, s_r$ and acts on $\hat{P}$ by

$$s_i \hat{\omega}_j = \hat{\omega}_j - \delta_{ij} \hat{\alpha}_i$$

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$$s_i \hat{\omega}_j = \hat{\omega}_j - \delta_{ij} \hat{\alpha}_i$$
- For a given $\hat{\mu} \in \hat{P}^k$, there exists a unique element $\hat{\lambda} \in \hat{P}^k$ such that $w(\hat{\mu} + \hat{\rho}) = \hat{\lambda} + \hat{\rho}$ for some $w \in \hat{W}$, $(\lambda + \rho | \alpha_a) \geq 0, \forall a \in I$ and $(\lambda + \rho | \theta) \leq k + h\}$.

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- Let $\hat{P}^k_+ := \{\sum_{i=0}^r \lambda_i\hat{\omega}_i \in \hat{P}^k | \lambda_i \geq 0\}$
- For $\hat{\lambda} \in \hat{P}^+_k$, $\mathcal{D}_{\hat{\lambda}} > 0$.

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- $\mathcal{D}_{w\cdot\hat{\lambda}} = (-1)^{\ell(w)}\mathcal{D}_{\hat{\lambda}}$ where $w \cdot \hat{\lambda} := w(\hat{\lambda} + \hat{\rho}) - \hat{\rho}$

affine Weyl group

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- $\mathcal{D}_{w\cdot\hat{\lambda}} = (-1)^{\ell(w)}\mathcal{D}_{\hat{\lambda}}$ where $w \cdot \hat{\lambda} := w(\hat{\lambda} + \hat{\rho}) - \hat{\rho}$
- If $w \in \hat{W}$ is an element of odd signature and $w \cdot \hat{\lambda} = \hat{\lambda}$, then $\mathcal{D}_{\hat{\lambda}} = 0$.

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WZW fusion ring

Definition

The WZW fusion ring is a free $\mathbb{Z}$-module equipped with the basis $\{V_{\hat{\omega}} | \hat{\omega} \in \hat{P}^k_+\}$ and the product is given by

$$V_{\hat{\lambda}} \cdot V_{\hat{\mu}} = \sum_{\hat{\nu} \in \hat{P}^k_+} N^{\hat{\nu}}_{\hat{\lambda}\hat{\mu}} V_{\hat{\nu}}$$

where the fusion coefficient $N^{\hat{\nu}}_{\hat{\lambda}\hat{\mu}}$ can be computed by the Verlinde formula

$$N^{\hat{\nu}}_{\hat{\lambda}\hat{\mu}} = \sum_{\hat{\omega} \in \hat{P}^k_+} \frac{S_{\hat{\lambda},\hat{\omega}} S_{\hat{\mu},\hat{\omega}} \overline{S_{\hat{\nu},\hat{\omega}}}}{S_{\hat{0},\hat{\omega}}}.$$

- commutative and associative, $V_{k\hat{\omega}_0}$ is the unity
- there exists an involution $V^*_{\hat{\omega}} := V_{\hat{\omega}^*}$

modular S-matrix and its properties

Definition

For a pair of weights $\hat{\lambda}, \hat{\mu} \in P_k$, we consider the quantity

$$S_{\hat{\lambda},\hat{\mu}} = C \sum_{w \in W} (-1)^{\ell(w)} \exp\left(-\frac{2\pi i (w(\lambda+\rho)|\mu+\rho)}{k+h}\right)$$

where $C = \frac{i^{|\Delta_+|}}{\sqrt{|P/Q^\vee|(k+h)^r}}$ is a normalizing factor independent of $\hat{\lambda}$ and $\hat{\mu}$.

- $S_{\hat{\lambda}\hat{\mu}} = S_{\hat{\mu}\hat{\lambda}}$.
- $S_{w\cdot\hat{\lambda},\hat{\mu}} = (-1)^{\ell(w)} S_{\hat{\lambda},\hat{\mu}}$ for $w \in \hat{W}$
- $S_{A\hat{\lambda},\hat{\mu}} = S_{\hat{\lambda},\hat{\mu}} e^{-2\pi i (A\omega_0|\mu)}$ (A : diagram automorphism)
- $S_{\hat{\lambda}^*,\hat{\mu}} = \overline{S_{\hat{\lambda},\hat{\mu}}}$ where $\lambda^* := -w_0\lambda \in P$ and w_0 is the longest element of the finite Weyl group W.

lifting up the KNS conjecture to the fusion ring

Definition

For each (a, m), we define an element $W_m^{(a)}$ (by abusing notation) of the fusion ring by

$$W_m^{(a)} := \sum_{\lambda \in P_+} Z(a, m, \lambda) V_{\hat{\lambda}}.$$

where $Z(a, m, \lambda)$ is the multiplicity of V_λ in the KR-module $\mathrm{res}\, W_m^{(a)}(u)$ as a $U_q(\mathfrak{g})$-module.

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where $Z(a, m, \lambda)$ is the multiplicity of V_λ in the KR-module $\mathrm{res}\, W_m^{(a)}(u)$ as a $U_q(\mathfrak{g})$-module.

Q: What will happen if we write $W_m^{(a)}$ as a linear combination of the basis of the fusion ring?

example : $X = A_3, k = 3$

This is easy since the decomposition is simple.

$$\begin{bmatrix} W_0^{(1)} & W_0^{(2)} & W_0^{(3)} \\ W_1^{(1)} & W_1^{(2)} & W_1^{(3)} \\ W_2^{(1)} & W_2^{(2)} & W_2^{(3)} \\ W_3^{(1)} & W_3^{(2)} & W_3^{(3)} \\ W_4^{(1)} & W_4^{(2)} & W_4^{(3)} \\ W_5^{(1)} & W_5^{(2)} & W_5^{(3)} \\ W_6^{(1)} & W_6^{(2)} & W_6^{(3)} \\ W_7^{(1)} & W_7^{(2)} & W_7^{(3)} \\ \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} V_{3\hat{\omega}_0} & V_{3\hat{\omega}_0} & V_{3\hat{\omega}_0} \\ V_{2\hat{\omega}_0+\hat{\omega}_1} & V_{2\hat{\omega}_0+\hat{\omega}_2} & V_{2\hat{\omega}_0+\hat{\omega}_3} \\ V_{\hat{\omega}_0+2\hat{\omega}_1} & V_{\hat{\omega}_0+2\hat{\omega}_2} & V_{\hat{\omega}_0+2\hat{\omega}_3} \\ V_{3\hat{\omega}_1} & V_{3\hat{\omega}_2} & V_{3\hat{\omega}_3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -V_{3\hat{\omega}_1} & V_{3\hat{\omega}_2} & -V_{3\hat{\omega}_3} \\ \vdots & \vdots & \vdots \end{bmatrix}$$

example : $X = D_5, k = 4$

It's easy for $a = 1, 4, 5$:

$$\begin{bmatrix} W_0^{(1)} & W_0^{(4)} & W_0^{(5)} \\ W_1^{(1)} & W_1^{(4)} & W_1^{(5)} \\ W_2^{(1)} & W_2^{(4)} & W_2^{(5)} \\ W_3^{(1)} & W_3^{(4)} & W_3^{(5)} \\ W_4^{(1)} & W_4^{(4)} & W_4^{(5)} \\ W_5^{(1)} & W_5^{(4)} & W_5^{(5)} \\ W_6^{(1)} & W_6^{(4)} & W_6^{(5)} \\ W_7^{(1)} & W_7^{(4)} & W_7^{(5)} \\ W_8^{(1)} & W_8^{(4)} & W_8^{(5)} \\ W_9^{(1)} & W_9^{(4)} & W_9^{(5)} \\ W_{10}^{(1)} & W_{10}^{(4)} & W_{10}^{(5)} \\ W_{11}^{(1)} & W_{11}^{(4)} & W_{11}^{(5)} \\ W_{12}^{(1)} & W_{12}^{(4)} & W_{12}^{(5)} \end{bmatrix} = \begin{bmatrix} V_{4\hat{\omega}_0} & V_{4\hat{\omega}_0} & V_{4\hat{\omega}_0} \\ V_{3\hat{\omega}_0+\hat{\omega}_1} & V_{3\hat{\omega}_0+\hat{\omega}_4} & V_{3\hat{\omega}_0+\hat{\omega}_5} \\ V_{2\hat{\omega}_0+2\hat{\omega}_1} & V_{2\hat{\omega}_0+2\hat{\omega}_4} & V_{2\hat{\omega}_0+2\hat{\omega}_5} \\ V_{\hat{\omega}_0+3\hat{\omega}_1} & V_{\hat{\omega}_0+3\hat{\omega}_4} & V_{\hat{\omega}_0+3\hat{\omega}_5} \\ V_{4\hat{\omega}_1} & V_{4\hat{\omega}_4} & V_{4\hat{\omega}_5} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ V_{4\hat{\omega}_1} & V_{4\hat{\omega}_4} & V_{4\hat{\omega}_5} \end{bmatrix}.$$

example : $X = D_5, k = 4$

For $a = 2$ and $0 \le m \le 4$,

$$\begin{bmatrix} W_0^{(2)} \\ W_1^{(2)} \\ W_2^{(2)} \\ W_3^{(2)} \\ W_4^{(2)} \end{bmatrix} = \begin{bmatrix} V_{4\hat{\omega}_0} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} + V_{2\hat{\omega}_2} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} + V_{2\hat{\omega}_2} + V_{3\hat{\omega}_2-2\hat{\omega}_0} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} + V_{2\hat{\omega}_2} + V_{-2\hat{\omega}_0+3\hat{\omega}_2} + V_{-4\hat{\omega}_0+4\hat{\omega}_2} \end{bmatrix}$$

Let us simplify

$$W_4^{(2)} = V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} + V_{2\hat{\omega}_2} + V_{-2\hat{\omega}_0+3\hat{\omega}_2} + V_{-4\hat{\omega}_0+4\hat{\omega}_2}.$$

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The shifted action of the affine Weyl group gives

$$s_0 \cdot (-2\hat{\omega}_0 + 3\hat{\omega}_2) = 2\hat{\omega}_2,$$
$$s_0 \cdot (-4\hat{\omega}_0 + 4\hat{\omega}_2) = 2\hat{\omega}_0 + \hat{\omega}_2.$$

example : $X = D_5, k = 4$

For $a = 2$ and $0 \leq m \leq 4$,

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Let us simplify

$$W_4^{(2)} = V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} + V_{2\hat{\omega}_2} + V_{-2\hat{\omega}_0+3\hat{\omega}_2} + V_{-4\hat{\omega}_0+4\hat{\omega}_2}.$$

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Thus $W_4^{(2)} = V_{4\hat{\omega}_0}$.

example : $X = D_5, k = 4$

Thus, for $a = 2$ and $0 \leq m \leq 4$,

$$\begin{bmatrix} W_0^{(2)} \\ W_1^{(2)} \\ W_2^{(2)} \\ W_3^{(2)} \\ W_4^{(2)} \end{bmatrix} = \begin{bmatrix} V_{4\hat{\omega}_0} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} + V_{2\hat{\omega}_2} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} + V_{2\hat{\omega}_2} + V_{3\hat{\omega}_2-2\hat{\omega}_0} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} + V_{2\hat{\omega}_2} + V_{-2\hat{\omega}_0+3\hat{\omega}_2} + V_{-4\hat{\omega}_0+4\hat{\omega}_2} \end{bmatrix}$$

example : $X = D_5, k = 4$

Thus, for $a = 2$ and $0 \leq m \leq 4$,

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$$= \begin{bmatrix} V_{4\hat{\omega}_0} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_2} + V_{2\hat{\omega}_0+\hat{\omega}_2} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} \\ V_{4\hat{\omega}_0} \end{bmatrix}$$

example : $X = D_5, k = 4$

For $a = 2, 3$ and $0 \le m \le h + k = 12$,

$$\begin{bmatrix} W_0^{(2)} & W_0^{(3)} \\ W_1^{(2)} & W_1^{(3)} \\ W_2^{(2)} & W_2^{(3)} \\ W_3^{(2)} & W_3^{(3)} \\ W_4^{(2)} & W_4^{(3)} \\ W_5^{(2)} & W_5^{(3)} \\ W_6^{(2)} & W_6^{(3)} \\ W_7^{(2)} & W_7^{(3)} \\ W_8^{(2)} & W_8^{(3)} \\ W_9^{(2)} & W_9^{(3)} \\ W_{10}^{(2)} & W_{10}^{(3)} \\ W_{11}^{(2)} & W_{11}^{(3)} \\ W_{12}^{(2)} & W_{12}^{(3)} \end{bmatrix} = \begin{bmatrix} V_{4\hat{\omega}_0} & V_{4\hat{\omega}_0} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} & V_{3\hat{\omega}_0+\hat{\omega}_1} + V_{2\hat{\omega}_0+\hat{\omega}_3} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_2} + V_{2\hat{\omega}_0+\hat{\omega}_2} & V_{2\hat{\omega}_0+2\hat{\omega}_1} + V_{2\hat{\omega}_3} + V_{\hat{\omega}_0+\hat{\omega}_1+\hat{\omega}_3} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} & V_{\hat{\omega}_0+3\hat{\omega}_1} + V_{2\hat{\omega}_1+\hat{\omega}_3} \\ V_{4\hat{\omega}_0} & V_{4\hat{\omega}_1} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ V_{4\hat{\omega}_0} & V_{4\hat{\omega}_1} \end{bmatrix}.$$

boundary of Q-system

Lemma

Let $(\tau_a)_{a \in I}$ *be as follows :*

	τ_a
A_r	$\hat{\omega}_a$
D_r	$\begin{cases} \hat{\omega}_1 & \text{if } 1 \leq a \leq r-2 \text{ and } a \equiv 1 \pmod 2 \\ \hat{\omega}_0 & \text{if } 1 \leq a \leq r-2 \text{ and } a \equiv 0 \pmod 2 \\ \hat{\omega}_a & \text{if } a = r-1 \text{ or } a = r \end{cases}$
$\vdots$	$\vdots$

Then $(V_{k\tau_a})_{a \in I}$ *satisfies the system of equations*

$$\left(Q^{(a)}\right)^2 = \prod_{b \sim a} Q^{(b)}, \quad a \in I.$$

main theorem : lifting of the KNS conjecture

Theorem (L '13)

For each $a \in I$, let us define $R_m^{(a)}$ in the fusion ring by

$$R_m^{(a)} = \begin{cases} W_m^{(a)} & 0 \leq m \leq \lfloor \frac{k}{2} \rfloor \\ V_{k\tau_a}(W_{k-m}^{(a)*}) & \lfloor \frac{k+1}{2} \rfloor \leq m \leq k \end{cases}$$

and $R_{-1}^{(a)} = R_{k+1}^{(a)} = 0$. Then $\left(R_m^{(a)}\right)$ is a positive solution of the level k restricted Q-system in the fusion ring.

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and $R_{-1}^{(a)} = R_{k+1}^{(a)} = 0$. Then $\left(R_m^{(a)}\right)$ is a positive solution of the level k restricted Q-system in the fusion ring.

- this holds for all classical types.

main theorem : lifting of the KNS conjecture

Theorem (L '13)

For each $a \in I$, let us define $R_m^{(a)}$ in the fusion ring by

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- this holds for all classical types.
- the KNS conjecture follows as a corollary (positivity, symmetry, unit boundary condition, ...)

problems : positivity and periodicity

Conjecture

For $a \in I$, let τ_a as before and $\sigma_a = e^{-2\pi i(\tau_a|\rho)}$. The following properties hold :

1. $W_m^{(a)}$ *is positive for* $0 \leq m \leq t_a k$,
2. $W_{k-m}^{(a)} = V_{k\tau_a}(W_m^{(a)*})$ *for* $0 \leq m \leq k$,
3. $W_k^{(a)} = V_{k\tau_a}$,
4. $W_{k+1}^{(a)} = W_{k+2}^{(a)} = \cdots = W_{(k+h^\vee)-1}^{(a)} = 0$,
5. $W_{m+n(k+h^\vee)}^{(a)} = \sigma_a^n V_{k\tau_a}^n W_m^{(a)}$ *for* $0 \leq m \leq (k+h^\vee)-1$ *and* $n \in \mathbb{Z}_{\geq 0}$.

completely verified only for type A_r

problems

- Study $\mathrm{Rep}(U_q(\hat{\mathfrak{g}})) \to \mathrm{Rep}(U_q(\mathfrak{g})) \to \mathrm{Fus}_k(\hat{\mathfrak{g}})$ for q generic

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for (X, X') pair of Cartan matrices? (Q-system corresponds to (X, A_{k-1}))

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- Thank you!