

# Kirillov-Reshetikhin modules and fusion rings from CFT

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# Q-system : definition

## Definition

Let  $X$  be a Dynkin diagram of type  $ADE$ . For a family of variables  $\{Q_m^{(a)} \mid a \in I, m \in \mathbb{Z}_{\geq 0}\}$ , consider recurrences given by

$$\left(Q_m^{(a)}\right)^2 = \prod_{b \in I, b \sim a} Q_m^{(b)} + Q_{m-1}^{(a)} Q_{m+1}^{(a)}$$

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Dynkin diagram :  $\bullet_{(1)} - \bullet_{(2)} - \bullet_{(3)} - \bullet_{(4)}$

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- Question : Fix a positive integer  $k \geq 1$  (level). How can we find  $(Q_1^{(a)})_{a \in I}$  such that  $Q_k^{(a)} = 1$  for all  $a \in I$ ?

level  $k$  restricted Q-system

## Definition

For variables  $(Q_m^{(a)})$  with  $0 \leq m \leq k$  and  $a \in I$ , consider a system of equations

$$\begin{cases} Q_0^{(a)} = 1 & a \in I \\ (Q_m^{(a)})^2 = \prod_{b \sim a} (Q_m^{(b)}) + Q_{m-1}^{(a)} Q_{m+1}^{(a)} & 1 \leq m \leq k-1, a \in I \\ Q_k^{(a)} = 1 & a \in I \end{cases}$$

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We call this system of equations the level  $k$  restricted Q-system.

- there exists a unique positive solution of the level  $k$  restricted Q-system.

# Rogers dilogarithm function

The Rogers dilogarithm function is defined by

$$L(x) = -\frac{1}{2} \int_0^x \frac{\log(1-y)}{y} + \frac{\log(y)}{1-y} dy$$

for  $x \in (0, 1)$ . We set  $L(0) = 0$  and  $L(1) = \pi^2/6$  so that  $L$  is continuous on  $[0, 1]$ .

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$$L(x) + L(1-xy) + L(y) + L\left(\frac{1-y}{1-xy}\right) + L\left(\frac{1-x}{1-xy}\right) = \frac{\pi^2}{2} = 3L(1)$$

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- shows up in many places (e.g. number theory, algebraic K-theory, hyperbolic geometry)
- computes central charges for some conformal field theories



# TBA and dilogarithm identities for conformal field theories

For variables  $\{f_m^{(a)} \mid a \in I, 1 \leq m \leq k-1\}$ , consider a system of equations given by

$$\sum_{b \in I} \mathcal{C}(X)_{ab} \log(1 - f_m^{(b)}) = \sum_{n=1}^{k-1} \mathcal{C}(A_{k-1})_{mn} \log f_n^{(a)}.$$

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**Theorem (Bazhanov, Kirillov, Reshetikhin '87, ..., Nakanishi '10)**

*Let  $X$  be a Dynkin diagram of type ADE of rank  $r$  and  $\mathfrak{g}$  be the corresponding simple Lie algebra. Let  $(f_m^{(a)})$  be the unique positive solution of the above such that  $0 < f_m^{(a)} < 1$ . Then*

$$\frac{6}{\pi^2} \sum_{a \in I} \sum_{m=1}^{k-1} L(f_m^{(a)}) = \frac{k \dim \mathfrak{g}}{h+k} - r = \frac{(k-1)hr}{h+k}$$

*where  $h$  denotes the Coxeter number of  $X$ .*

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$$\sum_{b \in I} \mathcal{C}(X)_{ab} \log(1 - f_m^{(b)}) = \sum_{n \in I'} \mathcal{C}(A_{k-1})_{mn} \log f_n^{(a)},$$

such that  $0 < f_m^{(a)} < 1$  can be constructed from the unique positive solution of level  $k$  restricted Q-system :

$$f_m^{(a)} = 1 - \frac{Q_{m-1}^{(a)} Q_{m+1}^{(a)}}{(Q_m^{(a)})^2} = \frac{\prod_{b \sim a} Q_m^{(b)}}{(Q_m^{(a)})^2}.$$

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- The quantized universal enveloping algebra  $U_q(\mathfrak{g})$  is contained in  $U_q(\hat{\mathfrak{g}})$  as a subalgebra
- lifting of an irreducible  $U_q(\mathfrak{g})$ -module of highest weight  $m\omega_a$  to an  $U_q(\hat{\mathfrak{g}})$ -module by adding more irreducible  $U_q(\mathfrak{g})$ -modules (minimal affinization)

# Kirillov-Reshetikhin (KR) modules

- For given  $\mathfrak{g}$ , KR module  $W_m^{(a)}(u)$  can be parametrized by  $a \in I$ ,  $m \in \mathbb{Z}_{\geq 0}$ , and  $u \in \mathbb{C}^\times$  spectral parameter

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## Theorem (Nakajima '03, Hernandez '06)

The classical characters  $\{Q_m^{(a)} \mid a \in I, m \in \mathbb{Z}_{\geq 0}\}$  satisfy the unrestricted Q-system :

$$\left(Q_m^{(a)}\right)^2 = \prod_{b \sim a} Q_m^{(b)} + Q_{m-1}^{(a)} Q_{m+1}^{(a)}.$$

## character solutions of the unrestricted Q-systems

For  $X = A_r$ , we have  $Q_m^{(a)} = \chi_{m\omega_a}$  for all  $a \in I$  and  $m \in \mathbb{Z}_{\geq 0}$

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For  $X = D_r$ ,

$$Q_m^{(a)} = \begin{cases} \sum \chi_{k_a\omega_a + k_{a-2}\omega_{a-2} + \cdots + k_1\omega_1} & 1 \leq a < r-1, a \equiv 1 \pmod{2}, \\ \sum \chi_{k_a\omega_a + k_{a-2}\omega_{a-2} + \cdots + k_0\omega_0} & 1 \leq a < r-1, a \equiv 0 \pmod{2}, \\ \chi_{m\omega_a} & a = r-1, r \end{cases}$$

where  $\omega_0 = 0$  and the summation is over all nonnegative integers satisfying  $k_a + k_{a-2} + \cdots + k_1 = m$  for  $a$  odd and  $k_a + k_{a-2} + \cdots + k_0 = m$  for  $a$  even.

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All known for classical types and partially known for exceptional types



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- Among those elements which one satisfies  $Q_m^{(a)} > 0$  for all  $0 \leq m \leq k$  and  $a \in I$ ?

# Kuniba-Nakanishi-Suzuki (KNS) conjecture

Let  $\rho = \sum_{i=1}^r \omega_i \in P$  the Weyl vector.

Conjecture (Kirillov '87, Kuniba-Nakanishi-Suzuki '92)

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- (unimodality)  $\mathcal{D}_{m-1}^{(a)} < \mathcal{D}_m^{(a)}$  holds true for  $1 \leq m \leq \lfloor \frac{k}{2} \rfloor$  and  $a \in I$  where  $\lfloor x \rfloor$  is the floor function.

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It has been known to be true for  $X = A_r$ .

numerical example :  $X = D_5, k = 4$

$$\begin{bmatrix} 1.000 & 1.000 & 1.000 & 1.000 & 1.000 \\ 3.732 & 8.464 & 14.93 & 4.732 & 4.732 \\ 5.464 & 15.93 & 33.32 & 7.464 & 7.464 \\ 3.732 & 8.464 & 14.93 & 4.732 & 4.732 \\ 1.000 & 1.000 & 1.000 & 1.000 & 1.000 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1.000 & 1.000 & 1.000 & 1.000 & 1.000 \\ 3.732 & 8.464 & 14.93 & 4.732 & 4.732 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

# from finite to affine by adding an extra vertex

For a given classical weight  $\lambda = \sum_{i=1}^r \lambda_i \omega_i \in P$ , define its level  $k$  affinization  $\hat{\lambda} = \sum_{i=0}^r \lambda_i \hat{\omega}_i \in \hat{P}_k$  where

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- $\hat{P}^k = \{\sum_{i=0}^r \lambda_i \hat{\omega}_i \in \hat{P} \mid \sum_{i=0}^r c_i \lambda_i = k\}$  where  $\theta = \sum_{i=1}^r c_i \alpha_i$  is the highest root and  $c_0 = 1$ .

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Example ( $X = D_5, k = 4$ )

$$\mathcal{D}_4^{(2)} = \chi_0 + \chi_{\omega_2} + \chi_{2\omega_2} + \chi_{3\omega_2} + \chi_{4\omega_2} \text{ evaluated at } \frac{\rho}{h+k}$$

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## Example ( $X = D_5, k = 4$ )

$$\begin{aligned} \mathcal{D}_4^{(2)} &= \chi_0 + \chi_{\omega_2} + \chi_{2\omega_2} + \chi_{3\omega_2} + \chi_{4\omega_2} \text{ evaluated at } \frac{\rho}{h+k} \\ &= \mathcal{D}_{4\hat{\omega}_0} + \mathcal{D}_{2\hat{\omega}_0 + \hat{\omega}_2} + \mathcal{D}_{2\hat{\omega}_2} + \mathcal{D}_{-2\hat{\omega}_0 + 3\hat{\omega}_2} + \mathcal{D}_{-4\hat{\omega}_0 + 4\hat{\omega}_2} \\ &= 1? \end{aligned}$$

For  $\hat{\lambda} \in \hat{P}_k$ ,  $\mathcal{D}_{\hat{\lambda}}$  is not positive in general!

# quantum dimensions

## Definition

For  $\hat{\lambda} \in \hat{P}_k$ , the quantum dimension or  $q$ -dimension of  $\hat{\lambda}$  is defined by

$$\mathcal{D}_{\hat{\lambda}} = \chi_{\lambda} \left( \frac{\rho}{h+k} \right) = \frac{\prod_{\alpha > 0} \sin \frac{\pi(\lambda + \rho|\alpha)}{h+k}}{\prod_{\alpha > 0} \sin \frac{\pi(\rho|\alpha)}{h+k}}$$

where  $(\cdot|\cdot)$  is the standard bilinear form on  $P$  such that  $(\theta|\theta) = 2$ .



# affine Weyl group

- The affine Weyl group  $\hat{W}$  is generated by  $s_0, s_1, \dots, s_r$  and acts on  $\hat{P}$  by

$$s_i \hat{\omega}_j = \hat{\omega}_j - \delta_{ij} \hat{\alpha}_i$$

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- For a given  $\hat{\mu} \in \hat{P}^k$ , there exists a unique element  $\hat{\lambda} \in \hat{P}^k$  such that  $w(\hat{\mu} + \hat{\rho}) = \hat{\lambda} + \hat{\rho}$  for some  $w \in \hat{W}$ ,  $(\lambda + \rho | \alpha_a) \geq 0, \forall a \in I$  and  $(\lambda + \rho | \theta) \leq k + h$ .

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- Let  $\hat{P}_+^k := \{\sum_{i=0}^r \lambda_i \hat{\omega}_i \in \hat{P}^k | \lambda_i \geq 0\}$
- For  $\hat{\lambda} \in \hat{P}_k^+$ ,  $\mathcal{D}_{\hat{\lambda}} > 0$ .

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- If  $w \in \hat{W}$  is an element of odd signature and  $w \cdot \hat{\lambda} = \hat{\lambda}$ ,  $\mathcal{D}_{\hat{\lambda}} = 0$ .

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# WZW fusion ring

## Definition

The WZW fusion ring is a free  $\mathbb{Z}$ -module equipped with the basis  $\{V_{\hat{\omega}} | \hat{\omega} \in \hat{P}_+^k\}$  and the product is defined by

$$V_{\hat{\lambda}} \cdot V_{\hat{\mu}} = \sum_{\hat{\nu} \in \hat{P}_+^k} N_{\hat{\lambda}\hat{\mu}}^{\hat{\nu}} V_{\hat{\nu}}$$

where the fusion coefficient  $N_{\hat{\lambda}\hat{\mu}}^{\hat{\nu}}$  can be computed by the Verlinde formula

$$N_{\hat{\lambda}\hat{\mu}}^{\hat{\nu}} = \sum_{\hat{\omega} \in \hat{P}_+^k} \frac{S_{\hat{\lambda}, \hat{\omega}} S_{\hat{\mu}, \hat{\omega}} \overline{S_{\hat{\nu}, \hat{\omega}}}}{S_{\hat{0}, \hat{\omega}}}.$$

- commutative and associative,  $V_{k\hat{\omega}_0}$  is the unity
- there exists an involution  $V_{\hat{\omega}}^* := V_{\hat{\omega}^*}$

## modular S-matrix and its properties

## Definition

For a pair of weights  $\hat{\lambda}, \hat{\mu} \in P_k$ , we consider the quantity

$$S_{\hat{\lambda}, \hat{\mu}} = C \sum_{w \in W} (-1)^{\ell(w)} \exp \left( -\frac{2\pi i (w(\lambda + \rho) | \mu + \rho)}{k + h} \right)$$

where  $C = \frac{i^{|\Delta_+|}}{\sqrt{|P/Q^\vee|(k+h)^r}}$  is a normalizing factor independent of  $\hat{\lambda}$  and  $\hat{\mu}$ .

- $S_{\hat{\lambda}\hat{\mu}} = S_{\hat{\mu}\hat{\lambda}}$ .
- $S_{w \cdot \hat{\lambda}, \hat{\mu}} = (-1)^{\ell(w)} S_{\hat{\lambda}, \hat{\mu}}$  for  $w \in \hat{W}$
- $S_{A\hat{\lambda}, \hat{\mu}} = S_{\hat{\lambda}, \hat{\mu}} e^{-2\pi i (A\omega_0 | \mu)}$  ( $A$  : diagram automorphism)
- $S_{\hat{\lambda}^*, \hat{\mu}} = \overline{S_{\hat{\lambda}, \hat{\mu}}}$  where  $\lambda^* := -w_0\lambda \in P$  and  $w_0$  is the longest element of the finite Weyl group  $W$ .



# lifting up the KNS conjecture to the fusion ring

## Definition

For each  $(a, m)$ , we define an element  $W_m^{(a)}$  (by abusing notation) of the fusion ring by

$$W_m^{(a)} := \sum_{\lambda \in P_+} Z(a, m, \lambda) V_{\hat{\lambda}}.$$

where  $Z(a, m, \lambda)$  is the multiplicity of  $V_{\lambda}$  in the KR-module  $\text{res } W_m^{(a)}(u)$  as a  $U_q(\mathfrak{g})$ -module.

Q: What will happen if we write  $W_m^{(a)}$  as a linear combination of the basis of the fusion ring?

example :  $X = A_3, k = 3$

This is easy since the decomposition is simple.

$$\begin{bmatrix}
 W_0^{(1)} & W_0^{(2)} & W_0^{(3)} \\
 W_1^{(1)} & W_1^{(2)} & W_1^{(3)} \\
 W_2^{(1)} & W_2^{(2)} & W_2^{(3)} \\
 W_3^{(1)} & W_3^{(2)} & W_3^{(3)} \\
 W_4^{(1)} & W_4^{(2)} & W_4^{(3)} \\
 W_5^{(1)} & W_5^{(2)} & W_5^{(3)} \\
 W_6^{(1)} & W_6^{(2)} & W_6^{(3)} \\
 W_7^{(1)} & W_7^{(2)} & W_7^{(3)} \\
 \vdots & \vdots & \vdots
 \end{bmatrix}
 =
 \begin{bmatrix}
 V_{3\hat{\omega}_0} & V_{3\hat{\omega}_0} & V_{3\hat{\omega}_0} \\
 V_{2\hat{\omega}_0+\hat{\omega}_1} & V_{2\hat{\omega}_0+\hat{\omega}_2} & V_{2\hat{\omega}_0+\hat{\omega}_3} \\
 V_{\hat{\omega}_0+2\hat{\omega}_1} & V_{\hat{\omega}_0+2\hat{\omega}_2} & V_{\hat{\omega}_0+2\hat{\omega}_3} \\
 V_{3\hat{\omega}_1} & V_{3\hat{\omega}_2} & V_{3\hat{\omega}_3} \\
 0 & 0 & 0 \\
 0 & 0 & 0 \\
 0 & 0 & 0 \\
 -V_{3\hat{\omega}_1} & V_{3\hat{\omega}_2} & -V_{3\hat{\omega}_3} \\
 \vdots & \vdots & \vdots
 \end{bmatrix}$$

example :  $X = D_5, k = 4$

It's easy for  $a = 1, 4, 5$  and  $0 \leq m \leq h + k = 12$ ,

$$\begin{bmatrix}
 W_0^{(1)} & W_0^{(4)} & W_0^{(5)} \\
 W_1^{(1)} & W_1^{(4)} & W_1^{(5)} \\
 W_2^{(1)} & W_2^{(4)} & W_2^{(5)} \\
 W_3^{(1)} & W_3^{(4)} & W_3^{(5)} \\
 W_4^{(1)} & W_4^{(4)} & W_4^{(5)} \\
 W_5^{(1)} & W_5^{(4)} & W_5^{(5)} \\
 W_6^{(1)} & W_6^{(4)} & W_6^{(5)} \\
 W_7^{(1)} & W_7^{(4)} & W_7^{(5)} \\
 W_8^{(1)} & W_8^{(4)} & W_8^{(5)} \\
 W_9^{(1)} & W_9^{(4)} & W_9^{(5)} \\
 W_{10}^{(1)} & W_{10}^{(4)} & W_{10}^{(5)} \\
 W_{11}^{(1)} & W_{11}^{(4)} & W_{11}^{(5)} \\
 W_{12}^{(1)} & W_{12}^{(4)} & W_{12}^{(5)}
 \end{bmatrix}
 =
 \begin{bmatrix}
 V_{4\hat{\omega}_0} & V_{4\hat{\omega}_0} & V_{4\hat{\omega}_0} \\
 V_{3\hat{\omega}_0+\hat{\omega}_1} & V_{3\hat{\omega}_0+\hat{\omega}_4} & V_{3\hat{\omega}_0+\hat{\omega}_5} \\
 V_{2\hat{\omega}_0+2\hat{\omega}_1} & V_{2\hat{\omega}_0+2\hat{\omega}_4} & V_{2\hat{\omega}_0+2\hat{\omega}_5} \\
 V_{\hat{\omega}_0+3\hat{\omega}_1} & V_{\hat{\omega}_0+3\hat{\omega}_4} & V_{\hat{\omega}_0+3\hat{\omega}_5} \\
 V_{4\hat{\omega}_1} & V_{4\hat{\omega}_4} & V_{4\hat{\omega}_5} \\
 0 & 0 & 0 \\
 0 & 0 & 0 \\
 0 & 0 & 0 \\
 0 & 0 & 0 \\
 0 & 0 & 0 \\
 0 & 0 & 0 \\
 0 & 0 & 0 \\
 0 & 0 & 0 \\
 V_{4\hat{\omega}_1} & V_{4\hat{\omega}_4} & V_{4\hat{\omega}_5}
 \end{bmatrix}
 .$$

example :  $X = D_5, k = 4$

For  $a = 2$  and  $0 \leq m \leq 4$ ,

$$\begin{bmatrix} W_0^{(2)} \\ W_1^{(2)} \\ W_2^{(2)} \\ W_3^{(2)} \\ W_4^{(2)} \end{bmatrix} = \begin{bmatrix} & & & & V_{4\hat{\omega}_0} \\ & & & & V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0 + \hat{\omega}_2} \\ & & & & V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0 + \hat{\omega}_2} + V_{2\hat{\omega}_2} \\ & & & & V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0 + \hat{\omega}_2} + V_{2\hat{\omega}_2} + V_{3\hat{\omega}_2 - 2\hat{\omega}_0} \\ & & & & V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0 + \hat{\omega}_2} + V_{2\hat{\omega}_2} + V_{-2\hat{\omega}_0 + 3\hat{\omega}_2} + V_{-4\hat{\omega}_0 + 4\hat{\omega}_2} \end{bmatrix}$$

Let us simplify

$$W_4^{(2)} = V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0 + \hat{\omega}_2} + V_{2\hat{\omega}_2} + V_{-2\hat{\omega}_0 + 3\hat{\omega}_2} + V_{-4\hat{\omega}_0 + 4\hat{\omega}_2}.$$

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The shifted action of the affine Weyl group gives

$$s_0 \cdot (-2\hat{\omega}_0 + 3\hat{\omega}_2) = 2\hat{\omega}_2,$$

$$s_0 \cdot (-4\hat{\omega}_0 + 4\hat{\omega}_2) = 2\hat{\omega}_0 + \hat{\omega}_2.$$





example :  $X = D_5, k = 4$

$$\begin{bmatrix} W_0^{(2)} \\ W_1^{(2)} \\ W_2^{(2)} \\ W_3^{(2)} \\ W_4^{(2)} \end{bmatrix} = \begin{bmatrix} & & & & V_{4\hat{\omega}_0} \\ & & & & \\ & & & V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0 + \hat{\omega}_2} & \\ & & V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0 + \hat{\omega}_2} + V_{2\hat{\omega}_2} & & \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0 + \hat{\omega}_2} + V_{2\hat{\omega}_2} + V_{3\hat{\omega}_2 - 2\hat{\omega}_0} & & & & \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0 + \hat{\omega}_2} + V_{2\hat{\omega}_2} + V_{-2\hat{\omega}_0 + 3\hat{\omega}_2} + V_{-4\hat{\omega}_0 + 4\hat{\omega}_2} & & & & \end{bmatrix}$$

$$= \begin{bmatrix} & & & & V_{4\hat{\omega}_0} \\ & & & & \\ & & & V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0 + \hat{\omega}_2} & \\ & & V_{4\hat{\omega}_0} + V_{2\hat{\omega}_2} + V_{2\hat{\omega}_0 + \hat{\omega}_2} & & \\ & V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0 + \hat{\omega}_2} & & & \\ & & V_{4\hat{\omega}_0} & & \end{bmatrix}$$





## boundary of Q-system

## Lemma

Let  $(\tau_a)_{a \in I}$  be as follows :

	$\tau_a$
$A_r$	$\hat{\omega}_a$
$D_r$	$\begin{cases} \hat{\omega}_1 & \text{if } 1 \leq a \leq r-2 \text{ and } a \equiv 1 \pmod{2} \\ \hat{\omega}_0 & \text{if } 1 \leq a \leq r-2 \text{ and } a \equiv 0 \pmod{2} \\ \hat{\omega}_a & \text{if } a = r-1 \text{ or } a = r \end{cases}$
$\vdots$	$\vdots$

Then  $(V_{k\tau_a})_{a \in I}$  satisfies the system of equations

$$\left(Q^{(a)}\right)^2 = \prod_{b \sim a} Q^{(b)}, \quad a \in I.$$

## main theorem : lifting of the KNS conjecture

## Theorem (L '13)

For each  $a \in I$ , let us define  $R_m^{(a)}$  in the fusion ring by

$$R_m^{(a)} = \begin{cases} W_m^{(a)} & 0 \leq m \leq \lfloor \frac{k}{2} \rfloor \\ V_{k\tau_a}(W_{k-m}^{(a)*}) & \lfloor \frac{k+1}{2} \rfloor \leq m \leq k \end{cases}$$

and  $R_{-1}^{(a)} = R_{k+1}^{(a)} = 0$ . Then  $(R_m^{(a)})$  is a positive solution of the level  $k$  restricted  $Q$ -system in the fusion ring.

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and  $R_{-1}^{(a)} = R_{k+1}^{(a)} = 0$ . Then  $(R_m^{(a)})$  is a positive solution of the level  $k$  restricted  $Q$ -system in the fusion ring.

- this holds for all classical types.
- the KNS conjecture follows as a corollary (positivity, symmetry, unit boundary condition, ...)

## problems

## Conjecture

For  $a \in I$ , let  $\tau_a$  as before and  $\sigma_a = e^{-2\pi i(\tau_a|\rho)}$ . The following properties hold :

- 1  $W_m^{(a)}$  is positive for  $0 \leq m \leq t_a k$ ,
- 2  $W_{k-m}^{(a)} = V_{k\tau_a}(W_m^{(a)*})$  for  $0 \leq m \leq k$ ,
- 3  $W_k^{(a)} = V_{k\tau_a}$ ,
- 4  $W_{k+1}^{(a)} = W_{k+2}^{(a)} = \dots = W_{(k+h^\vee)-1}^{(a)} = 0$ ,
- 5  $W_{m+n(k+h^\vee)}^{(a)} = \sigma_a^n V_{k\tau_a}^n W_m^{(a)}$  for  $0 \leq m \leq (k+h^\vee) - 1$  and  $n \in \mathbb{Z}_{\geq 0}$ .

# problems

- representation theoretic interpretation is missing
- Kirillov-Reshetikhin modules for  $q$  roots of unity
- Fermionic formula for fusion ring (for generic  $q$ , KR modules give fermionic formulas for  $\mathfrak{g}$ )
- what can we do for

$$z_{ii'}^2 = \prod_{j \in I} z_{ji'}^{\mathcal{I}(X)_{ij}} + \prod_{j' \in I'} z_{ij'}^{\mathcal{I}(X')_{i'j'}}$$

for  $(X, X')$  pair of Cartan matrices? (Q-system corresponds to  $(X, A_{k-1})$ )

- Thank you!