# Kirillov-Reshetikhin modules and fusion rings from CFT

Chul-hee Lee

MPI

Seminar on Algebra, Geometry and Physics, 9/7/2013

# Q-system : definition

## Definition

Let $X$ be a Dynkin diagram of type $ADE$. For a family of variables $\{Q_m^{(a)} | a \in I, m \in \mathbb{Z}_{\geq 0}\}$, consider recurrences given by

$$\left(Q_m^{(a)}\right)^2 = \prod_{b \in I, b \sim a} Q_m^{(b)} + Q_{m-1}^{(a)} Q_{m+1}^{(a)}$$

# Q-system : definition

## Definition

Let $X$ be a Dynkin diagram of type $ADE$. For a family of variables $\{Q_m^{(a)} | a \in I, m \in \mathbb{Z}_{\geq 0}\}$, consider recurrences given by

$$\left(Q_m^{(a)}\right)^2 = \prod_{b \in I, b \sim a} Q_m^{(b)} + Q_{m-1}^{(a)} Q_{m+1}^{(a)}$$

We call this system the unrestricted $Q$-system of type $X$. We use boundary conditions $Q_0^{(a)} = 1$ for all $a \in I$.

# Q-system : definition

## Definition

Let $X$ be a Dynkin diagram of type $ADE$. For a family of variables $\{Q_m^{(a)} | a \in I, m \in \mathbb{Z}_{\geq 0}\}$, consider recurrences given by

$$\left(Q_m^{(a)}\right)^2 = \prod_{b \in I, b \sim a} Q_m^{(b)} + Q_{m-1}^{(a)} Q_{m+1}^{(a)}$$

We call this system the unrestricted $Q$-system of type $X$. We use boundary conditions $Q_0^{(a)} = 1$ for all $a \in I$.

# Q-system : $A_4$ example

Dynkin diagram : $\bullet_{(1)} - \bullet_{(2)} - \bullet_{(3)} - \bullet_{(4)}$
Use the recursion

$$Q_{m+1}^{(a)} = \frac{\left(Q_m^{(a)}\right)^2 - \prod_{b\sim a} Q_m^{(b)}}{Q_{m-1}^{(a)}}$$

# Q-system : $A_4$ example

Dynkin diagram : $\bullet_{(1)} - \bullet_{(2)} - \bullet_{(3)} - \bullet_{(4)}$
Use the recursion

$$Q^{(a)}_{m+1} = \frac{\left(Q^{(a)}_m\right)^2 - \prod_{b\sim a} Q^{(b)}_m}{Q^{(a)}_{m-1}}$$

| $m \backslash a$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| 1 | $Q^{(1)}_1$ | $Q^{(2)}_1$ | $Q^{(3)}_1$ | $Q^{(4)}_1$ |

# Q-system : $A_4$ example

Dynkin diagram : $\bullet_{(1)} - \bullet_{(2)} - \bullet_{(3)} - \bullet_{(4)}$
Use the recursion

$$Q^{(a)}_{m+1} = \frac{\left(Q^{(a)}_m\right)^2 - \prod_{b\sim a} Q^{(b)}_m}{Q^{(a)}_{m-1}}$$

| $m\backslash a$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| 1 | $Q^{(1)}_1$ | $Q^{(2)}_1$ | $Q^{(3)}_1$ | $Q^{(4)}_1$ |
| 2 | $Q^{(1)}_2$ | $Q^{(2)}_2$ | $Q^{(3)}_2$ | $Q^{(4)}_2$ |

# Q-system : $A_4$ example

Dynkin diagram : $\bullet_{(1)} - \bullet_{(2)} - \bullet_{(3)} - \bullet_{(4)}$
Use the recursion

$$Q_{m+1}^{(a)} = \frac{\left(Q_m^{(a)}\right)^2 - \prod_{b\sim a} Q_m^{(b)}}{Q_{m-1}^{(a)}}$$

| $m \backslash a$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| 1 | $Q_1^{(1)}$ | $Q_1^{(2)}$ | $Q_1^{(3)}$ | $Q_1^{(4)}$ |
| 2 | $Q_2^{(1)}$ | $Q_2^{(2)}$ | $Q_2^{(3)}$ | $Q_2^{(4)}$ |
| 3 | $Q_3^{(1)}$ | $Q_3^{(2)}$ | $Q_3^{(3)}$ | $Q_3^{(4)}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

# Q-system : $A_4$ example

Dynkin diagram : $\bullet_{(1)} - \bullet_{(2)} - \bullet_{(3)} - \bullet_{(4)}$

Use the recursion

$$Q^{(a)}_{m+1} = \frac{\left(Q^{(a)}_m\right)^2 - \prod_{b\sim a} Q^{(b)}_m}{Q^{(a)}_{m-1}}$$

| $m\backslash a$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| 1 | $Q^{(1)}_1$ | $Q^{(2)}_1$ | $Q^{(3)}_1$ | $Q^{(4)}_1$ |
| 2 | $Q^{(1)}_2$ | $Q^{(2)}_2$ | $Q^{(3)}_2$ | $Q^{(4)}_2$ |
| 3 | $Q^{(1)}_3$ | $Q^{(2)}_3$ | $Q^{(3)}_3$ | $Q^{(4)}_3$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

- Question : Fix a positive integer $k \geq 1$ (level). How can we find $(Q^{(a)}_1)_{a\in I}$ such that $Q^{(a)}_k = 1$ for all $a \in I$?

# level $k$ restricted Q-system

## Definition

For variables $\left(Q_m^{(a)}\right)$ with $0 \le m \le k$ and $a \in I$, consider a system of equations

$$\begin{cases} Q_0^{(a)} = 1 & a \in I \\ \left(Q_m^{(a)}\right)^2 = \prod_{b \sim a} \left(Q_m^{(b)}\right) + Q_{m-1}^{(a)} Q_{m+1}^{(a)} & 1 \le m \le k-1, a \in I \\ Q_k^{(a)} = 1 & a \in I \end{cases}$$

# level $k$ restricted Q-system

## Definition

For variables $\left(Q_m^{(a)}\right)$ with $0 \leq m \leq k$ and $a \in I$, consider a system of equations

$$\begin{cases} Q_0^{(a)} = 1 & a \in I \\ \left(Q_m^{(a)}\right)^2 = \prod_{b \sim a} \left(Q_m^{(b)}\right) + Q_{m-1}^{(a)} Q_{m+1}^{(a)} & 1 \leq m \leq k-1, a \in I \\ Q_k^{(a)} = 1 & a \in I \end{cases}$$

We call this system of equations the level $k$ restricted $Q$-system.

# level $k$ restricted Q-system

**Definition**

For variables $\left(Q_m^{(a)}\right)$ with $0 \leq m \leq k$ and $a \in I$, consider a system of equations

$$\begin{cases} Q_0^{(a)} = 1 & a \in I \\ \left(Q_m^{(a)}\right)^2 = \prod_{b \sim a} \left(Q_m^{(b)}\right) + Q_{m-1}^{(a)} Q_{m+1}^{(a)} & 1 \leq m \leq k-1, a \in I \\ Q_k^{(a)} = 1 & a \in I \end{cases}$$

We call this system of equations the level $k$ restricted $Q$-system.

- there exists a unique positive solution of the level $k$ restricted $Q$-system.

# Rogers dilogarithm function

The Rogers dilogarithm function is defined by

$$L(x) = -\frac{1}{2}\int_0^x \frac{\log(1-y)}{y} + \frac{\log(y)}{1-y} dy$$

for $x \in (0,1)$. We set $L(0) = 0$ and $L(1) = \pi^2/6$ so that $L$ is continuous on $[0,1]$.

# Rogers dilogarithm function

The Rogers dilogarithm function is defined by

$$L(x) = -\frac{1}{2}\int_0^x \frac{\log(1-y)}{y} + \frac{\log(y)}{1-y} dy$$

for $x \in (0,1)$. We set $L(0) = 0$ and $L(1) = \pi^2/6$ so that $L$ is continuous on $[0,1]$.

- many functional identities are satisfied. For example,

$$L(x)+L(1-xy)+L(y)+L(\frac{1-y}{1-xy})+L\left(\frac{1-x}{1-xy}\right) = \frac{\pi^2}{2} = 3L(1)$$

# Rogers dilogarithm function

The Rogers dilogarithm function is defined by

$$L(x) = -\frac{1}{2}\int_0^x \frac{\log(1-y)}{y} + \frac{\log(y)}{1-y}dy$$

for $x \in (0,1)$. We set $L(0) = 0$ and $L(1) = \pi^2/6$ so that $L$ is continuous on $[0,1]$.

- many functional identities are satisfied. For example,

$$L(x)+L(1-xy)+L(y)+L(\frac{1-y}{1-xy})+L\left(\frac{1-x}{1-xy}\right) = \frac{\pi^2}{2} = 3L(1)$$

- shows up in many places (e.g. number theory, algebraic K-theory, hyperbolic geometry)

# Rogers dilogarithm function

The Rogers dilogarithm function is defined by

$$L(x) = -\frac{1}{2}\int_0^x \frac{\log(1-y)}{y} + \frac{\log(y)}{1-y} dy$$

for $x \in (0,1)$. We set $L(0) = 0$ and $L(1) = \pi^2/6$ so that $L$ is continuous on $[0,1]$.

- many functional identities are satisfied. For example,

$$L(x)+L(1-xy)+L(y)+L(\frac{1-y}{1-xy})+L\left(\frac{1-x}{1-xy}\right) = \frac{\pi^2}{2} = 3L(1)$$

- shows up in many places (e.g. number theory, algebraic K-theory, hyperbolic geometry)
- computes central charges for some conformal field theories

# TBA and dilogarithm identities for conformal field theories

For variables $\{f_m^{(a)} | a \in I, 1 \leq m \leq k-1\}$, consider a system of equations given by

$$\sum_{b \in I} \mathcal{C}(X)_{ab} \log(1 - f_m^{(b)}) = \sum_{n \in I'} \mathcal{C}(A_{k-1})_{mn} \log f_n^{(a)}.$$

# TBA and dilogarithm identities for conformal field theories

For variables $\{f_m^{(a)} | a \in I, 1 \leq m \leq k-1\}$, consider a system of equations given by

$$\sum_{b \in I} \mathcal{C}(X)_{ab} \log(1 - f_m^{(b)}) = \sum_{n \in I'} \mathcal{C}(A_{k-1})_{mn} \log f_n^{(a)}.$$

**Theorem (Bazhanov,Kirillov,Reshetikhin '87,$\cdots$, Nakanishi '10)**

*Let $X$ be a Dynkin diagram of type ADE of rank $r$ and $\mathfrak{g}$ be the corresponding simple Lie algebra. Let $(f_m^{(a)})$ be the unique positive solution of the above such that $0 < f_m^{(a)} < 1$. Then*

$$\frac{6}{\pi^2} \sum_{a \in I} \sum_{m=1}^{k-1} L(f_m^{(a)}) = \frac{k \dim \mathfrak{g}}{h+k} - \operatorname{rank} \mathfrak{g} = \frac{(k-1)hr}{h+k}$$

*where $h$ denotes the Coxeter number of $X$.*

# TBA and dilogarithm identities for conformal field theories

For variables $\{f_m^{(a)} | a \in I, 1 \leq m \leq k-1\}$, consider a system of equations given by

$$\sum_{b \in I} \mathcal{C}(X)_{ab} \log(1 - f_m^{(b)}) = \sum_{n \in I'} \mathcal{C}(A_{k-1})_{mn} \log f_n^{(a)}.$$

**Theorem (Bazhanov,Kirillov,Reshetikhin '87,$\cdots$, Nakanishi '10)**

*Let $X$ be a Dynkin diagram of type ADE of rank $r$ and $\mathfrak{g}$ be the corresponding simple Lie algebra. Let $(f_m^{(a)})$ be the unique positive solution of the above such that $0 < f_m^{(a)} < 1$. Then*

$$\frac{6}{\pi^2} \sum_{a \in I} \sum_{m=1}^{k-1} L(f_m^{(a)}) = \frac{k \dim \mathfrak{g}}{h+k} - \operatorname{rank} \mathfrak{g} = \frac{(k-1)hr}{h+k}$$

*where $h$ denotes the Coxeter number of $X$.*

# TBA and dilogarithm identities for conformal field theories

- Originated from the Thermodynamic Bethe ansatz (TBA) calculations for some models in statistical mechanics and had been open for many years

# TBA and dilogarithm identities for conformal field theories

- Originated from the Thermodynamic Bethe ansatz (TBA) calculations for some models in statistical mechanics and had been open for many years
- The unique positive solution of the equations

$$\sum_{b\in I} \mathcal{C}(X)_{ab} \log(1 - f_m^{(b)}) = \sum_{n\in I'} \mathcal{C}(A_{k-1})_{mn} \log f_n^{(a)},$$

  such that $0 < f_m^{(a)} < 1$ can be constructed from the unique positive solution of level $k$ restricted $Q$-system :

$$f_m^{(a)} = 1 - \frac{Q_{m-1}^{(a)} Q_{m+1}^{(a)}}{(Q_m^{(a)})^2} = \frac{\prod_{b\sim a} Q_m^{(b)}}{(Q_m^{(a)})^2}.$$

# Kirillov-Reshetikhin (KR) modules

- Let $q$ be a non-zero complex number which is not a root of unity

# Kirillov-Reshetikhin (KR) modules

- Let $q$ be a non-zero complex number which is not a root of unity
- KR modules form a special class of finite dimensional modules of the quantum affine algebra $U_q(\hat{\mathfrak{g}})$

# Kirillov-Reshetikhin (KR) modules

- Let $q$ be a non-zero complex number which is not a root of unity
- KR modules form a special class of finite dimensional modules of the quantum affine algebra $U_q(\hat{\mathfrak{g}})$
- The quantized universal enveloping algebra $U_q(\mathfrak{g})$ is contained in $U_q(\hat{\mathfrak{g}})$ as a subalgebra

# Kirillov-Reshetikhin (KR) modules

- Let $q$ be a non-zero complex number which is not a root of unity
- KR modules form a special class of finite dimensional modules of the quantum affine algebra $U_q(\hat{\mathfrak{g}})$
- The quantized universal enveloping algebra $U_q(\mathfrak{g})$ is contained in $U_q(\hat{\mathfrak{g}})$ as a subalgebra
- lifting of an irreducible $U_q(\mathfrak{g})$-module of highest weight $m\omega_a$ to an $U_q(\hat{\mathfrak{g}})$-module by adding more irreducible $U_q(\mathfrak{g})$-modules (minimal affinization)

# Kirillov-Reshetikhin (KR) modules

- For given $\mathfrak{g}$, KR module $W_m^{(a)}(u)$ can be parametrized by $a \in I$, $m \in \mathbb{Z}_{\geq 0}$, and $u \in \mathbb{C}^{\times}$ spectral parameter

# Kirillov-Reshetikhin (KR) modules

- For given $\mathfrak{g}$, KR module $W_m^{(a)}(u)$ can be parametrized by $a \in I$, $m \in \mathbb{Z}_{\geq 0}$, and $u \in \mathbb{C}^\times$ spectral parameter
- From $U_q(\hat{\mathfrak{g}})$-module $W_m^{(a)}(u)$, we can get a finite dimensional $U_q(\mathfrak{g})$-module $\mathrm{res}\, W_m^{(a)}(u)$ by restriction

# Kirillov-Reshetikhin (KR) modules

- For given $\mathfrak{g}$, KR module $W_m^{(a)}(u)$ can be parametrized by $a \in I$, $m \in \mathbb{Z}_{\geq 0}$, and $u \in \mathbb{C}^\times$ spectral parameter
- From $U_q(\hat{\mathfrak{g}})$-module $W_m^{(a)}(u)$, we can get a finite dimensional $U_q(\mathfrak{g})$-module $\mathrm{res}\, W_m^{(a)}(u)$ by restriction
- Let $Q_m^{(a)}$ be the classical character of $\mathrm{res}\, W_m^{(a)}(u)$ and then $Q_m^{(a)}$ can be written as $\sum_{\lambda \in P_+} Z(a, m, \lambda)\chi_\lambda$ for some $Z(a, m, \lambda) \in \mathbb{Z}_{\geq 0}$ where $\chi_\lambda$ is the character of a highest weight representation

# Kirillov-Reshetikhin (KR) modules

- For given $\mathfrak{g}$, KR module $W_m^{(a)}(u)$ can be parametrized by $a \in I$, $m \in \mathbb{Z}_{\geq 0}$, and $u \in \mathbb{C}^\times$ spectral parameter
- From $U_q(\hat{\mathfrak{g}})$-module $W_m^{(a)}(u)$, we can get a finite dimensional $U_q(\mathfrak{g})$-module $\mathrm{res}\, W_m^{(a)}(u)$ by restriction
- Let $Q_m^{(a)}$ be the classical character of $\mathrm{res}\, W_m^{(a)}(u)$ and then $Q_m^{(a)}$ can be written as $\sum_{\lambda \in P_+} Z(a, m, \lambda)\chi_\lambda$ for some $Z(a, m, \lambda) \in \mathbb{Z}_{\geq 0}$ where $\chi_\lambda$ is the character of a highest weight representation

**Theorem (Nakajima '03, Hernandez '06)**

*The classical characters $\{Q_m^{(a)} | a \in I, m \in \mathbb{Z}_{\geq 0}\}$ satisfy the unrestricted Q-system :*

$$\left(Q_m^{(a)}\right)^2 = \prod_{b \sim a} Q_m^{(a)} + Q_{m-1}^{(a)} Q_{m+1}^{(a)}.$$

# character solutions of the unrestricted Q-systems

For $X = A_r$, we have $Q_m^{(a)} = \chi_{m\omega_a}$ for all $a \in I$ and $m \in \mathbb{Z}_{\geq 0}$

# character solutions of the unrestricted Q-systems

For $X = A_r$, we have $Q^{(a)}_m = \chi_{m\omega_a}$ for all $a \in I$ and $m \in \mathbb{Z}_{\geq 0}$
For $X = D_r$,

$$Q^{(a)}_m = \begin{cases} \sum \chi_{k_a\omega_a + k_{a-2}\omega_{a-2}+\cdots+k_1\omega_1} & 1 \leq a < r-1, a \equiv 1 \pmod 2, \\ \sum \chi_{k_a\omega_a + k_{a-2}\omega_{a-2}+\cdots+k_0\omega_0} & 1 \leq a < r-1, a \equiv 0 \pmod 2, \\ \chi_{m\omega_a} & a = r-1, r \end{cases}$$

where $\omega_0 = 0$ and the summation is over all nonnegative integers satisfying $k_a + k_{a-2} + \cdots + k_1 = m$ for $a$ odd and $k_a + k_{a-2} + \cdots + k_0 = m$ for $a$ even.

# character solutions of the unrestricted Q-systems

For $X = A_r$, we have $Q_m^{(a)} = \chi_{m\omega_a}$ for all $a \in I$ and $m \in \mathbb{Z}_{\geq 0}$
For $X = D_r$,

$$Q_m^{(a)} = \begin{cases} \sum \chi_{k_a\omega_a + k_{a-2}\omega_{a-2} + \cdots + k_1\omega_1} & 1 \leq a < r-1, a \equiv 1 \pmod 2, \\ \sum \chi_{k_a\omega_a + k_{a-2}\omega_{a-2} + \cdots + k_0\omega_0} & 1 \leq a < r-1, a \equiv 0 \pmod 2, \\ \chi_{m\omega_a} & a = r-1, r \end{cases}$$

where $\omega_0 = 0$ and the summation is over all nonnegative integers satisfying $k_a + k_{a-2} + \cdots + k_1 = m$ for $a$ odd and $k_a + k_{a-2} + \cdots + k_0 = m$ for $a$ even.
All known for classical types and partially known for exceptional types

# using characters to solve the level $k$ restricted $Q$-system

- We can get a solution of unrestricted $Q$-system by specializing characters $Q_m^{(a)}$ at elements of $\mathfrak{h}$ or $\mathfrak{h}^*$

# using characters to solve the level $k$ restricted $Q$-system

- We can get a solution of unrestricted $Q$-system by specializing characters $Q_m^{(a)}$ at elements of $\mathfrak{h}$ or $\mathfrak{h}^*$
- For a given level $k \geq 1$, which elements can satisfy $Q_k^{(a)} = 1$ for all $a \in I$ ?

# using characters to solve the level $k$ restricted $Q$-system

- We can get a solution of unrestricted $Q$-system by specializing characters $Q_m^{(a)}$ at elements of $\mathfrak{h}$ or $\mathfrak{h}^*$
- For a given level $k \geq 1$, which elements can satisfy $Q_k^{(a)} = 1$ for all $a \in I$ ?
- Among those elements which one satisfies $Q_m^{(a)} > 0$ for all $0 \leq m \leq k$ and $a \in I$?

# Kuniba-Nakanishi-Suzuki (KNS) conjecture

Let $\rho = \sum_{i=1}^{r} \omega_i \in P$ the Weyl vector.

**Conjecture (Kirillov '87, Kuniba-Nakanishi-Suzuki '92)**

*Let $\mathcal{D}_m^{(a)} = Q_m^{(a)}(\frac{\rho}{h+k})$ for each $(a, m)$. It satisfies the following properties :*

# Kuniba-Nakanishi-Suzuki (KNS) conjecture

Let $\rho = \sum_{i=1}^{r} \omega_i \in P$ the Weyl vector.

**Conjecture (Kirillov '87, Kuniba-Nakanishi-Suzuki '92)**

*Let $\mathcal{D}_m^{(a)} = Q_m^{(a)}(\frac{\rho}{h+k})$ for each $(a, m)$. It satisfies the following properties :*

- *(positivity)* $\mathcal{D}_m^{(a)} > 0$ *for* $0 \leq m \leq k$ *and* $a \in I$.

# Kuniba-Nakanishi-Suzuki (KNS) conjecture

Let $\rho = \sum_{i=1}^{r} \omega_i \in P$ the Weyl vector.

**Conjecture (Kirillov '87, Kuniba-Nakanishi-Suzuki '92)**

*Let $\mathcal{D}_m^{(a)} = Q_m^{(a)}(\frac{\rho}{h+k})$ for each $(a, m)$. It satisfies the following properties :*

- *(positivity)* $\mathcal{D}_m^{(a)} > 0$ *for* $0 \leq m \leq k$ *and* $a \in I$.
- *(unit boundary condition)* $\mathcal{D}_k^{(a)} = 1$ *for* $a \in I$.

# Kuniba-Nakanishi-Suzuki (KNS) conjecture

Let $\rho = \sum_{i=1}^{r} \omega_i \in P$ the Weyl vector.

**Conjecture (Kirillov '87, Kuniba-Nakanishi-Suzuki '92)**

*Let $\mathcal{D}_m^{(a)} = Q_m^{(a)}(\frac{\rho}{h+k})$ for each $(a, m)$. It satisfies the following properties :*

- *(positivity)* $\mathcal{D}_m^{(a)} > 0$ *for* $0 \leq m \leq k$ *and* $a \in I$.
- *(unit boundary condition)* $\mathcal{D}_k^{(a)} = 1$ *for* $a \in I$.
- *(occurrence of 0)* $\mathcal{D}_{k+1}^{(a)} = \mathcal{D}_{k+2}^{(a)} = \cdots = \mathcal{D}_{k+h-1}^{(a)} = 0$ *for* $a \in I$.

# Kuniba-Nakanishi-Suzuki (KNS) conjecture

Let $\rho = \sum_{i=1}^{r} \omega_i \in P$ the Weyl vector.

**Conjecture (Kirillov '87, Kuniba-Nakanishi-Suzuki '92)**

*Let $\mathcal{D}_m^{(a)} = Q_m^{(a)}(\frac{\rho}{h+k})$ for each $(a, m)$. It satisfies the following properties :*

- *(positivity)* $\mathcal{D}_m^{(a)} > 0$ *for* $0 \leq m \leq k$ *and* $a \in I$.
- *(unit boundary condition)* $\mathcal{D}_k^{(a)} = 1$ *for* $a \in I$.
- *(occurrence of 0)* $\mathcal{D}_{k+1}^{(a)} = \mathcal{D}_{k+2}^{(a)} = \cdots = \mathcal{D}_{k+h-1}^{(a)} = 0$ *for* $a \in I$.
- *(symmetry)* $\mathcal{D}_m^{(a)} = \mathcal{D}_{k-m}^{(a)}$ *for* $1 \leq m \leq k-1$ *and* $a \in I$.

# Kuniba-Nakanishi-Suzuki (KNS) conjecture

Let $\rho = \sum_{i=1}^{r} \omega_i \in P$ the Weyl vector.

**Conjecture (Kirillov '87, Kuniba-Nakanishi-Suzuki '92)**

*Let $\mathcal{D}_m^{(a)} = Q_m^{(a)}(\frac{\rho}{h+k})$ for each $(a, m)$. It satisfies the following properties :*

- *(positivity)* $\mathcal{D}_m^{(a)} > 0$ *for* $0 \leq m \leq k$ *and* $a \in I$.
- *(unit boundary condition)* $\mathcal{D}_k^{(a)} = 1$ *for* $a \in I$.
- *(occurrence of 0)* $\mathcal{D}_{k+1}^{(a)} = \mathcal{D}_{k+2}^{(a)} = \cdots = \mathcal{D}_{k+h-1}^{(a)} = 0$ *for* $a \in I$.
- *(symmetry)* $\mathcal{D}_m^{(a)} = \mathcal{D}_{k-m}^{(a)}$ *for* $1 \leq m \leq k-1$ *and* $a \in I$.
- *(unimodality)* $\mathcal{D}_{m-1}^{(a)} < \mathcal{D}_m^{(a)}$ *holds true for* $1 \leq m \leq \lfloor \frac{k}{2} \rfloor$ *and* $a \in I$ *where* $\lfloor x \rfloor$ *is the floor function.*

# Kuniba-Nakanishi-Suzuki (KNS) conjecture

Let $\rho = \sum_{i=1}^{r} \omega_i \in P$ the Weyl vector.

**Conjecture (Kirillov '87, Kuniba-Nakanishi-Suzuki '92)**

*Let $\mathcal{D}_m^{(a)} = Q_m^{(a)}(\frac{\rho}{h+k})$ for each $(a, m)$. It satisfies the following properties :*

- *(positivity)* $\mathcal{D}_m^{(a)} > 0$ *for* $0 \leq m \leq k$ *and* $a \in I$.
- *(unit boundary condition)* $\mathcal{D}_k^{(a)} = 1$ *for* $a \in I$.
- *(occurrence of 0)* $\mathcal{D}_{k+1}^{(a)} = \mathcal{D}_{k+2}^{(a)} = \cdots = \mathcal{D}_{k+h-1}^{(a)} = 0$ *for* $a \in I$.
- *(symmetry)* $\mathcal{D}_m^{(a)} = \mathcal{D}_{k-m}^{(a)}$ *for* $1 \leq m \leq k-1$ *and* $a \in I$.
- *(unimodality)* $\mathcal{D}_{m-1}^{(a)} < \mathcal{D}_m^{(a)}$ *holds true for* $1 \leq m \leq \lfloor \frac{k}{2} \rfloor$ *and* $a \in I$ *where* $\lfloor x \rfloor$ *is the floor function.*

It has been known to be true for $X = A_r$.

# numerical example : $X = D_5, k = 4$

$$\begin{bmatrix} 1.000 & 1.000 & 1.000 & 1.000 & 1.000 \\ 3.732 & 8.464 & 14.93 & 4.732 & 4.732 \\ 5.464 & 15.93 & 33.32 & 7.464 & 7.464 \\ 3.732 & 8.464 & 14.93 & 4.732 & 4.732 \\ 1.000 & 1.000 & 1.000 & 1.000 & 1.000 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1.000 & 1.000 & 1.000 & 1.000 & 1.000 \\ 3.732 & 8.464 & 14.93 & 4.732 & 4.732 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

# from finite to affine by adding an extra vertex

For a given classical weight $\lambda = \sum_{i=1}^{r} \lambda_i \omega_i \in P$, define its level $k$ affinization $\hat{\lambda} = \sum_{i=0}^{r} \lambda_i \hat{\omega}_i \in \hat{P}_k$ where

- $\hat{P} = \mathbb{Z}\hat{\omega}_0 \oplus \cdots \oplus \mathbb{Z}\hat{\omega}_r$ the affine weight lattice

# from finite to affine by adding an extra vertex

For a given classical weight $\lambda = \sum_{i=1}^{r} \lambda_i \omega_i \in P$, define its level $k$ affinization $\hat{\lambda} = \sum_{i=0}^{r} \lambda_i \hat{\omega}_i \in \hat{P}_k$ where

- $\hat{P} = \mathbb{Z}\hat{\omega}_0 \oplus \cdots \oplus \mathbb{Z}\hat{\omega}_r$ the affine weight lattice
- $\hat{P}^k = \{\sum_{i=0}^{r} \lambda_i \hat{\omega}_i \in \hat{P} | \sum_{i=0}^{r} c_i \lambda_i = k\}$ where $\theta = \sum_{i=1}^{r} c_i \alpha_i$ is the highest root and $c_0 = 1$.

# from finite to affine by adding an extra vertex

For a given classical weight $\lambda = \sum_{i=1}^{r} \lambda_i \omega_i \in P$, define its level $k$ affinization $\hat{\lambda} = \sum_{i=0}^{r} \lambda_i \hat{\omega}_i \in \hat{P}_k$ where

- $\hat{P} = \mathbb{Z}\hat{\omega}_0 \oplus \cdots \oplus \mathbb{Z}\hat{\omega}_r$ the affine weight lattice
- $\hat{P}^k = \{\sum_{i=0}^{r} \lambda_i \hat{\omega}_i \in \hat{P} | \sum_{i=0}^{r} c_i \lambda_i = k\}$ where $\theta = \sum_{i=1}^{r} c_i \alpha_i$ is the highest root and $c_0 = 1$.

**Example ($X = D_5, k = 4$)**

$$\mathcal{D}_4^{(2)} = \chi_0 + \chi_{\omega_2} + \chi_{2\omega_2} + \chi_{3\omega_2} + \chi_{4\omega_2} \text{ evaluated at } \frac{\rho}{h+k}$$

# from finite to affine by adding an extra vertex

For a given classical weight $\lambda = \sum_{i=1}^{r} \lambda_i \omega_i \in P$, define its level $k$ affinization $\hat{\lambda} = \sum_{i=0}^{r} \lambda_i \hat{\omega}_i \in \hat{P}_k$ where

- $\hat{P} = \mathbb{Z}\hat{\omega}_0 \oplus \cdots \oplus \mathbb{Z}\hat{\omega}_r$ the affine weight lattice
- $\hat{P}^k = \{\sum_{i=0}^{r} \lambda_i \hat{\omega}_i \in \hat{P} | \sum_{i=0}^{r} c_i \lambda_i = k\}$ where $\theta = \sum_{i=1}^{r} c_i \alpha_i$ is the highest root and $c_0 = 1$.

**Example ($X = D_5, k = 4$)**

$$\begin{aligned}\mathcal{D}_4^{(2)} &= \chi_0 + \chi_{\omega_2} + \chi_{2\omega_2} + \chi_{3\omega_2} + \chi_{4\omega_2} \text{ evaluated at } \frac{\rho}{h+k} \\ &= \mathcal{D}_{4\hat{\omega}_0} + \mathcal{D}_{2\hat{\omega}_0+\hat{\omega}_2} + \mathcal{D}_{2\hat{\omega}_2} + \mathcal{D}_{-2\hat{\omega}_0+3\hat{\omega}_2} + \mathcal{D}_{-4\hat{\omega}_0+4\hat{\omega}_2} \\ &= 1?\end{aligned}$$

For $\hat{\lambda} \in \hat{P}_k$, $\mathcal{D}_{\hat{\lambda}}$ is not positive in general!

# quantum dimensions

## Definition

For $\hat{\lambda} \in \hat{P}_k$, the quantum dimension or $q$-dimension of $\hat{\lambda}$ is defined by

$$\mathcal{D}_{\hat{\lambda}} = \chi_\lambda\left(\frac{\rho}{h+k}\right) = \frac{\prod_{\alpha>0} \sin \frac{\pi(\lambda+\rho|\alpha)}{h+k}}{\prod_{\alpha>0} \sin \frac{\pi(\rho|\alpha)}{h+k}}$$

where $(\cdot|\cdot)$ is the standard bilinear form on $P$ such that $(\theta|\theta) = 2$.

# affine Weyl group

- The affine Weyl group $\hat{W}$ is generated by $s_0, s_1, \cdots, s_r$ and acts on $\hat{P}$ by
$$s_i\hat{\omega}_j = \hat{\omega}_j - \delta_{ij}\hat{\alpha}_i$$

# affine Weyl group

- The affine Weyl group $\hat{W}$ is generated by $s_0, s_1, \cdots, s_r$ and acts on $\hat{P}$ by
$$s_i\hat{\omega}_j = \hat{\omega}_j - \delta_{ij}\hat{\alpha}_i$$
- For a given $\hat{\mu} \in \hat{P}^k$, there exists a unique element $\hat{\lambda} \in \hat{P}^k$ such that $w(\hat{\mu} + \hat{\rho}) = \hat{\lambda} + \hat{\rho}$ for some $w \in \hat{W}$, $(\lambda + \rho|\alpha_a) \geq 0, \forall a \in I$ and $(\lambda + \rho|\theta) \leq k + h\}$.

# affine Weyl group

- The affine Weyl group $\hat{W}$ is generated by $s_0, s_1, \cdots, s_r$ and acts on $\hat{P}$ by
$$s_i \hat{\omega}_j = \hat{\omega}_j - \delta_{ij}\hat{\alpha}_i$$
- For a given $\hat{\mu} \in \hat{P}^k$, there exists a unique element $\hat{\lambda} \in \hat{P}^k$ such that $w(\hat{\mu} + \hat{\rho}) = \hat{\lambda} + \hat{\rho}$ for some $w \in \hat{W}$, $(\lambda + \rho|\alpha_a) \geq 0, \forall a \in I$ and $(\lambda + \rho|\theta) \leq k + h\}$.
- Let $\hat{P}^k_+ := \{\sum_{i=0}^r \lambda_i \hat{\omega}_i \in \hat{P}^k | \lambda_i \geq 0\}$
- For $\hat{\lambda} \in \hat{P}^+_k$, $\mathcal{D}_{\hat{\lambda}} > 0$.

# affine Weyl group

- The affine Weyl group $\hat{W}$ is generated by $s_0, s_1, \cdots, s_r$ and acts on $\hat{P}$ by
$$s_i \hat{\omega}_j = \hat{\omega}_j - \delta_{ij} \hat{\alpha}_i$$
- For a given $\hat{\mu} \in \hat{P}^k$, there exists a unique element $\hat{\lambda} \in \hat{P}^k$ such that $w(\hat{\mu} + \hat{\rho}) = \hat{\lambda} + \hat{\rho}$ for some $w \in \hat{W}$, $(\lambda + \rho | \alpha_a) \geq 0, \forall a \in I$ and $(\lambda + \rho | \theta) \leq k + h\}$.
- Let $\hat{P}^k_+ := \{\sum_{i=0}^r \lambda_i \hat{\omega}_i \in \hat{P}^k | \lambda_i \geq 0\}$
- For $\hat{\lambda} \in \hat{P}^+_k$, $\mathcal{D}_{\hat{\lambda}} > 0$.
- $\mathcal{D}_{w \cdot \hat{\lambda}} = (-1)^{\ell(w)} \mathcal{D}_{\hat{\lambda}}$ where $w \cdot \hat{\lambda} := w(\hat{\lambda} + \hat{\rho}) - \hat{\rho}$

# affine Weyl group

- The affine Weyl group $\hat{W}$ is generated by $s_0, s_1, \cdots, s_r$ and acts on $\hat{P}$ by
$$s_i\hat{\omega}_j = \hat{\omega}_j - \delta_{ij}\hat{\alpha}_i$$
- For a given $\hat{\mu} \in \hat{P}^k$, there exists a unique element $\hat{\lambda} \in \hat{P}^k$ such that $w(\hat{\mu} + \hat{\rho}) = \hat{\lambda} + \hat{\rho}$ for some $w \in \hat{W}$, $(\lambda + \rho|\alpha_a) \geq 0, \forall a \in I$ and $(\lambda + \rho|\theta) \leq k + h\}$.
- Let $\hat{P}^k_+ := \{\sum_{i=0}^r \lambda_i\hat{\omega}_i \in \hat{P}^k | \lambda_i \geq 0\}$
- For $\hat{\lambda} \in \hat{P}^+_k$, $\mathcal{D}_{\hat{\lambda}} > 0$.
- $\mathcal{D}_{w\cdot\hat{\lambda}} = (-1)^{\ell(w)}\mathcal{D}_{\hat{\lambda}}$ where $w \cdot \hat{\lambda} := w(\hat{\lambda} + \hat{\rho}) - \hat{\rho}$
- If $w \in \hat{W}$ is an element of odd signature and $w \cdot \hat{\lambda} = \hat{\lambda}$, $\mathcal{D}_{\hat{\lambda}} = 0$.

# affine Weyl group

- The affine Weyl group $\hat{W}$ is generated by $s_0, s_1, \cdots, s_r$ and acts on $\hat{P}$ by
$$s_i \hat{\omega}_j = \hat{\omega}_j - \delta_{ij}\hat{\alpha}_i$$
- For a given $\hat{\mu} \in \hat{P}^k$, there exists a unique element $\hat{\lambda} \in \hat{P}^k$ such that $w(\hat{\mu} + \hat{\rho}) = \hat{\lambda} + \hat{\rho}$ for some $w \in \hat{W}$, $(\lambda + \rho|\alpha_a) \geq 0, \forall a \in I$ and $(\lambda + \rho|\theta) \leq k + h\}$.
- Let $\hat{P}^k_+ := \{\sum_{i=0}^r \lambda_i \hat{\omega}_i \in \hat{P}^k | \lambda_i \geq 0\}$
- For $\hat{\lambda} \in \hat{P}_k^+$, $\mathcal{D}_{\hat{\lambda}} > 0$.
- $\mathcal{D}_{w\cdot\hat{\lambda}} = (-1)^{\ell(w)}\mathcal{D}_{\hat{\lambda}}$ where $w \cdot \hat{\lambda} := w(\hat{\lambda} + \hat{\rho}) - \hat{\rho}$
- If $w \in \hat{W}$ is an element of odd signature and $w \cdot \hat{\lambda} = \hat{\lambda}$, $\mathcal{D}_{\hat{\lambda}} = 0$.

# WZW fusion ring

## Definition

The WZW fusion ring is a free $\mathbb{Z}$-module equipped with the basis $\{V_{\hat{\omega}} | \hat{\omega} \in \hat{P}_+^k\}$ and the product is defined by

$$V_{\hat{\lambda}} \cdot V_{\hat{\mu}} = \sum_{\hat{\nu} \in \hat{P}_+^k} N_{\hat{\lambda}\hat{\mu}}^{\hat{\nu}} V_{\hat{\nu}}$$

where the fusion coefficient $N_{\hat{\lambda}\hat{\mu}}^{\hat{\nu}}$ can be computed by the Verlinde formula

$$N_{\hat{\lambda}\hat{\mu}}^{\hat{\nu}} = \sum_{\hat{\omega} \in \hat{P}_+^k} \frac{S_{\hat{\lambda},\hat{\omega}} S_{\hat{\mu},\hat{\omega}} \overline{S_{\hat{\nu},\hat{\omega}}}}{S_{\hat{0},\hat{\omega}}}.$$

- commutative and associative, $V_{k\hat{\omega}_0}$ is the unity
- there exists an involution $V_{\hat{\omega}}^* := V_{\hat{\omega}^*}$

# modular S-matrix and its properties

**Definition**

For a pair of weights $\hat{\lambda}, \hat{\mu} \in P_k$, we consider the quantity

$$S_{\hat{\lambda},\hat{\mu}} = C \sum_{w \in W} (-1)^{\ell(w)} \exp\left(-\frac{2\pi i (w(\lambda+\rho)|\mu+\rho)}{k+h}\right)$$

where $C = \frac{i^{|\Delta_+|}}{\sqrt{|P/Q^\vee|(k+h)^r}}$ is a normalizing factor independent of $\hat{\lambda}$ and $\hat{\mu}$.

- $S_{\hat{\lambda}\hat{\mu}} = S_{\hat{\mu}\hat{\lambda}}$.
- $S_{w\cdot\hat{\lambda},\hat{\mu}} = (-1)^{\ell(w)} S_{\hat{\lambda},\hat{\mu}}$ for $w \in \hat{W}$
- $S_{A\hat{\lambda},\hat{\mu}} = S_{\hat{\lambda},\hat{\mu}} e^{-2\pi i (A\omega_0|\mu)}$ ($A$ : diagram automorphism)
- $S_{\hat{\lambda}^*,\hat{\mu}} = \overline{S_{\hat{\lambda},\hat{\mu}}}$ where $\lambda^* := -w_0\lambda \in P$ and $w_0$ is the longest element of the finite Weyl group $W$.

# lifting up the KNS conjecture to the fusion ring

## Definition

For each $(a, m)$, we define an element $W_m^{(a)}$ (by abusing notation) of the fusion ring by

$$W_m^{(a)} := \sum_{\lambda \in P_+} Z(a, m, \lambda) V_{\hat{\lambda}}.$$

where $Z(a, m, \lambda)$ is the multiplicity of $V_\lambda$ in the KR-module $\mathrm{res}\, W_m^{(a)}(u)$ as a $U_q(\mathfrak{g})$-module.

Q: What will happen if we write $W_m^{(a)}$ as a linear combination of the basis of the fusion ring?

# example : $X = A_3, k = 3$

This is easy since the decomposition is simple.

$$\begin{bmatrix} W_0^{(1)} & W_0^{(2)} & W_0^{(3)} \\ W_1^{(1)} & W_1^{(2)} & W_1^{(3)} \\ W_2^{(1)} & W_2^{(2)} & W_2^{(3)} \\ W_3^{(1)} & W_3^{(2)} & W_3^{(3)} \\ W_4^{(1)} & W_4^{(2)} & W_4^{(3)} \\ W_5^{(1)} & W_5^{(2)} & W_5^{(3)} \\ W_6^{(1)} & W_6^{(2)} & W_6^{(3)} \\ W_7^{(1)} & W_7^{(2)} & W_7^{(3)} \\ \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} V_{3\hat{\omega}_0} & V_{3\hat{\omega}_0} & V_{3\hat{\omega}_0} \\ V_{2\hat{\omega}_0+\hat{\omega}_1} & V_{2\hat{\omega}_0+\hat{\omega}_2} & V_{2\hat{\omega}_0+\hat{\omega}_3} \\ V_{\hat{\omega}_0+2\hat{\omega}_1} & V_{\hat{\omega}_0+2\hat{\omega}_2} & V_{\hat{\omega}_0+2\hat{\omega}_3} \\ V_{3\hat{\omega}_1} & V_{3\hat{\omega}_2} & V_{3\hat{\omega}_3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -V_{3\hat{\omega}_1} & V_{3\hat{\omega}_2} & -V_{3\hat{\omega}_3} \\ \vdots & \vdots & \vdots \end{bmatrix}$$

# example : $X = D_5, k = 4$

It's easy for $a = 1, 4, 5$ and $0 \le m \le h + k = 12$,

$$\begin{bmatrix} W_0^{(1)} & W_0^{(4)} & W_0^{(5)} \\ W_1^{(1)} & W_1^{(4)} & W_1^{(5)} \\ W_2^{(1)} & W_2^{(4)} & W_2^{(5)} \\ W_3^{(1)} & W_3^{(4)} & W_3^{(5)} \\ W_4^{(1)} & W_4^{(4)} & W_4^{(5)} \\ W_5^{(1)} & W_5^{(4)} & W_5^{(5)} \\ W_6^{(1)} & W_6^{(4)} & W_6^{(5)} \\ W_7^{(1)} & W_7^{(4)} & W_7^{(5)} \\ W_8^{(1)} & W_8^{(4)} & W_8^{(5)} \\ W_9^{(1)} & W_9^{(4)} & W_9^{(5)} \\ W_{10}^{(1)} & W_{10}^{(4)} & W_{10}^{(5)} \\ W_{11}^{(1)} & W_{11}^{(4)} & W_{11}^{(5)} \\ W_{12}^{(1)} & W_{12}^{(4)} & W_{12}^{(5)} \end{bmatrix} = \begin{bmatrix} V_{4\hat\omega_0} & V_{4\hat\omega_0} & V_{4\hat\omega_0} \\ V_{3\hat\omega_0+\hat\omega_1} & V_{3\hat\omega_0+\hat\omega_4} & V_{3\hat\omega_0+\hat\omega_5} \\ V_{2\hat\omega_0+2\hat\omega_1} & V_{2\hat\omega_0+2\hat\omega_4} & V_{2\hat\omega_0+2\hat\omega_5} \\ V_{\hat\omega_0+3\hat\omega_1} & V_{\hat\omega_0+3\hat\omega_4} & V_{\hat\omega_0+3\hat\omega_5} \\ V_{4\hat\omega_1} & V_{4\hat\omega_4} & V_{4\hat\omega_5} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ V_{4\hat\omega_1} & V_{4\hat\omega_4} & V_{4\hat\omega_5} \end{bmatrix}.$$

# example : $X = D_5, k = 4$

For $a = 2$ and $0 \leq m \leq 4$,

$$\begin{bmatrix} W_0^{(2)} \\ W_1^{(2)} \\ W_2^{(2)} \\ W_3^{(2)} \\ W_4^{(2)} \end{bmatrix} = \begin{bmatrix} V_{4\hat{\omega}_0} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} + V_{2\hat{\omega}_2} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} + V_{2\hat{\omega}_2} + V_{3\hat{\omega}_2-2\hat{\omega}_0} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} + V_{2\hat{\omega}_2} + V_{-2\hat{\omega}_0+3\hat{\omega}_2} + V_{-4\hat{\omega}_0+4\hat{\omega}_2} \end{bmatrix}$$

Let us simplify

$$W_4^{(2)} = V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} + V_{2\hat{\omega}_2} + V_{-2\hat{\omega}_0+3\hat{\omega}_2} + V_{-4\hat{\omega}_0+4\hat{\omega}_2}.$$

# example : $X = D_5, k = 4$

For $a = 2$ and $0 \leq m \leq 4$,

$$\begin{bmatrix} W_0^{(2)} \\ W_1^{(2)} \\ W_2^{(2)} \\ W_3^{(2)} \\ W_4^{(2)} \end{bmatrix} = \begin{bmatrix} V_{4\hat{\omega}_0} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} + V_{2\hat{\omega}_2} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} + V_{2\hat{\omega}_2} + V_{3\hat{\omega}_2-2\hat{\omega}_0} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} + V_{2\hat{\omega}_2} + V_{-2\hat{\omega}_0+3\hat{\omega}_2} + V_{-4\hat{\omega}_0+4\hat{\omega}_2} \end{bmatrix}$$

Let us simplify

$$W_4^{(2)} = V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} + V_{2\hat{\omega}_2} + V_{-2\hat{\omega}_0+3\hat{\omega}_2} + V_{-4\hat{\omega}_0+4\hat{\omega}_2}.$$

The shifted action of the affine Weyl group gives

$$s_0 \cdot (-2\hat{\omega}_0 + 3\hat{\omega}_2) = 2\hat{\omega}_2,$$
$$s_0 \cdot (-4\hat{\omega}_0 + 4\hat{\omega}_2) = 2\hat{\omega}_0 + \hat{\omega}_2.$$

# example : $X = D_5, k = 4$

For $a = 2$ and $0 \leq m \leq 4$,

$$\begin{bmatrix} W_0^{(2)} \\ W_1^{(2)} \\ W_2^{(2)} \\ W_3^{(2)} \\ W_4^{(2)} \end{bmatrix} = \begin{bmatrix} V_{4\hat{\omega}_0} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} + V_{2\hat{\omega}_2} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} + V_{2\hat{\omega}_2} + V_{3\hat{\omega}_2-2\hat{\omega}_0} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} + V_{2\hat{\omega}_2} + V_{-2\hat{\omega}_0+3\hat{\omega}_2} + V_{-4\hat{\omega}_0+4\hat{\omega}_2} \end{bmatrix}$$

Let us simplify

$$W_4^{(2)} = V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} + V_{2\hat{\omega}_2} + V_{-2\hat{\omega}_0+3\hat{\omega}_2} + V_{-4\hat{\omega}_0+4\hat{\omega}_2}.$$

The shifted action of the affine Weyl group gives

$$s_0 \cdot (-2\hat{\omega}_0 + 3\hat{\omega}_2) = 2\hat{\omega}_2,$$
$$s_0 \cdot (-4\hat{\omega}_0 + 4\hat{\omega}_2) = 2\hat{\omega}_0 + \hat{\omega}_2.$$

Thus $W_4^{(2)} = V_{4\hat{\omega}_0}$.

# example : $X = D_5, k = 4$

$$\begin{bmatrix} W_0^{(2)} \\ W_1^{(2)} \\ W_2^{(2)} \\ W_3^{(2)} \\ W_4^{(2)} \end{bmatrix} = \begin{bmatrix} V_{4\hat{\omega}_0} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} + V_{2\hat{\omega}_2} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} + V_{2\hat{\omega}_2} + V_{3\hat{\omega}_2-2\hat{\omega}_0} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} + V_{2\hat{\omega}_2} + V_{-2\hat{\omega}_0+3\hat{\omega}_2} + V_{-4\hat{\omega}_0+4\hat{\omega}_2} \end{bmatrix}$$

# example : $X = D_5, k = 4$

$$\begin{bmatrix} W_0^{(2)} \\ W_1^{(2)} \\ W_2^{(2)} \\ W_3^{(2)} \\ W_4^{(2)} \end{bmatrix} = \begin{bmatrix} V_{4\hat{\omega}_0} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} + V_{2\hat{\omega}_2} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} + V_{2\hat{\omega}_2} + V_{3\hat{\omega}_2-2\hat{\omega}_0} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} + V_{2\hat{\omega}_2} + V_{-2\hat{\omega}_0+3\hat{\omega}_2} + V_{-4\hat{\omega}_0+4\hat{\omega}_2} \end{bmatrix}$$

$$= \begin{bmatrix} V_{4\hat{\omega}_0} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_2} + V_{2\hat{\omega}_0+\hat{\omega}_2} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} \\ V_{4\hat{\omega}_0} \end{bmatrix}$$

# example : $X = D_5, k = 4$

For $a = 2, 3$ and $1 \le m \le h + k = 12$,

$$
\begin{bmatrix}
W_0^{(2)} & W_0^{(3)} \\
W_1^{(2)} & W_1^{(3)} \\
W_2^{(2)} & W_2^{(3)} \\
W_3^{(2)} & W_3^{(3)} \\
W_4^{(2)} & W_4^{(3)} \\
W_5^{(2)} & W_5^{(3)} \\
W_6^{(2)} & W_6^{(3)} \\
W_7^{(2)} & W_7^{(3)} \\
W_8^{(2)} & W_8^{(3)} \\
W_9^{(2)} & W_9^{(3)} \\
W_{10}^{(2)} & W_{10}^{(3)} \\
W_{11}^{(2)} & W_{11}^{(3)} \\
W_{12}^{(2)} & W_{12}^{(3)}
\end{bmatrix}
=
\begin{bmatrix}
V_{4\hat\omega_0} & V_{4\hat\omega_0} \\
V_{4\hat\omega_0} + V_{2\hat\omega_0+\hat\omega_2} & V_{3\hat\omega_0+\hat\omega_1} + V_{2\hat\omega_0+\hat\omega_3} \\
V_{4\hat\omega_0} + V_{2\hat\omega_2} + V_{2\hat\omega_0+\hat\omega_2} & V_{2\hat\omega_0+2\hat\omega_1} + V_{2\hat\omega_3} + V_{\hat\omega_0+\hat\omega_1+\hat\omega_3} \\
V_{4\hat\omega_0} + V_{2\hat\omega_0+\hat\omega_2} & V_{\hat\omega_0+3\hat\omega_1} + V_{2\hat\omega_1+\hat\omega_3} \\
V_{4\hat\omega_0} & V_{4\hat\omega_1} \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
V_{4\hat\omega_0} & V_{4\hat\omega_1}
\end{bmatrix}.
$$

# boundary of Q-system

## Lemma

*Let $(\tau_a)_{a\in I}$ be as follows :*

| | $\tau_a$ |
|---|---|
| $A_r$ | $\hat{\omega}_a$ |
| $D_r$ | $\begin{cases} \hat{\omega}_1 & \textit{if } 1 \le a \le r-2 \textit{ and } a \equiv 1 \pmod 2 \\ \hat{\omega}_0 & \textit{if } 1 \le a \le r-2 \textit{ and } a \equiv 0 \pmod 2 \\ \hat{\omega}_a & \textit{if } a = r-1 \textit{ or } a = r \end{cases}$ |
| $\vdots$ | $\vdots$ |

*Then $(V_{k\tau_a})_{a\in I}$ satisfies the system of equations*

$$\left(Q^{(a)}\right)^2 = \prod_{b\sim a} Q^{(b)}, \quad a \in I.$$

# main theorem : lifting of the KNS conjecture

**Theorem (L '13)**

*For each $a \in I$, let us define $R_m^{(a)}$ in the fusion ring by*

$$R_m^{(a)} = \begin{cases} W_m^{(a)} & 0 \leq m \leq \lfloor \frac{k}{2} \rfloor \\ V_{k\tau_a}(W_{k-m}^{(a)*}) & \lfloor \frac{k+1}{2} \rfloor \leq m \leq k \end{cases}$$

*and $R_{-1}^{(a)} = R_{k+1}^{(a)} = 0$. Then $\left(R_m^{(a)}\right)$ is a positive solution of the level $k$ restricted Q-system in the fusion ring.*

# main theorem : lifting of the KNS conjecture

**Theorem (L '13)**

*For each $a \in I$, let us define $R_m^{(a)}$ in the fusion ring by*

$$R_m^{(a)} = \begin{cases} W_m^{(a)} & 0 \leq m \leq \lfloor \frac{k}{2} \rfloor \\ V_{k\tau_a}(W_{k-m}^{(a)*}) & \lfloor \frac{k+1}{2} \rfloor \leq m \leq k \end{cases}$$

*and $R_{-1}^{(a)} = R_{k+1}^{(a)} = 0$. Then $\left(R_m^{(a)}\right)$ is a positive solution of the level $k$ restricted Q-system in the fusion ring.*

- this holds for all classical types.

# main theorem : lifting of the KNS conjecture

**Theorem (L '13)**

*For each $a \in I$, let us define $R_m^{(a)}$ in the fusion ring by*

$$R_m^{(a)} = \begin{cases} W_m^{(a)} & 0 \le m \le \lfloor \frac{k}{2} \rfloor \\ V_{k\tau_a}(W_{k-m}^{(a)*}) & \lfloor \frac{k+1}{2} \rfloor \le m \le k \end{cases}$$

*and $R_{-1}^{(a)} = R_{k+1}^{(a)} = 0$. Then $\left(R_m^{(a)}\right)$ is a positive solution of the level $k$ restricted Q-system in the fusion ring.*

- this holds for all classical types.
- the KNS conjecture follows as a corollary (positivity, symmetry, unit boundary condition, ...)

# problems

**Conjecture**

*For $a \in I$, let $\tau_a$ as before and $\sigma_a = e^{-2\pi i(\tau_a|\rho)}$. The following properties hold :*

1. $W_m^{(a)}$ *is positive for* $0 \le m \le t_a k$,
2. $W_{k-m}^{(a)} = V_{k\tau_a}(W_m^{(a)*})$ *for* $0 \le m \le k$,
3. $W_k^{(a)} = V_{k\tau_a}$,
4. $W_{k+1}^{(a)} = W_{k+2}^{(a)} = \cdots = W_{(k+h^\vee)-1}^{(a)} = 0$,
5. $W_{m+n(k+h^\vee)}^{(a)} = \sigma_a^n V_{k\tau_a}^n W_m^{(a)}$ *for* $0 \le m \le (k+h^\vee)-1$ *and* $n \in \mathbb{Z}_{\ge 0}$.

## problems

- representation theoretic interpretation is missing
- Kirillov-Reshetikhin modules for $q$ roots of unity
- Fermionic formula for fusion ring (for generic $q$, KR modules give fermionic formulas for $\mathfrak{g}$)
- what can we do for

$$z_{ii'}^2 = \prod_{j \in I} z_{ji'}^{\mathcal{I}(X)_{ij}} + \prod_{j' \in I'} z_{ij'}^{\mathcal{I}(X')_{i'j'}}$$

  for $(X, X')$ pair of Cartan matrices? ($Q$-system corresponds to $(X, A_{k-1})$)
- Thank you!