Kirillov-Reshetikhin modules and the WZW fusion ring

Chul-hee Lee

Seoul National University

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overview

- Q-systems and level restricted Q-systems
- their relations to dilogarithm identities
- Kirillov-Reshetikhin modules
- a conjecture of Kirillov and Kuniba-Nakanishi-Suzuki on level restricted Q-systems
- WZW fusion rings
- resolution and reformulation of the conjecture using WZW fusion rings

Q-system : definition

Let X be a simply-laced Dynkin diagram and I be the set of its vertices.

Definition

For a family of variables $\{Q_m^{(a)} | a \in I, m \in \mathbb{Z}_{\geq 0}\}$ in a commutative ring (like $\mathbb{C}$), consider recurrences given by

$$\left(Q_m^{(a)}\right)^2 = \prod_{b \in I, b \sim a} Q_m^{(b)} + Q_{m-1}^{(a)} Q_{m+1}^{(a)}$$

where $\sim$ denotes the adjacency relation. We call this system the unrestricted Q-system of type X. We use boundary conditions $Q_0^{(a)} = 1$ for all $a \in I$.

- There is a more general and complicated definition of the Q-system associated to a multiply-laced Dynkin diagram

Q-system : A_4 example

Dynkin diagram : $\bullet_{(1)} - \bullet_{(2)} - \bullet_{(3)} - \bullet_{(4)}$
Use the recursion

$$Q^{(a)}_{m+1} = \frac{\left(Q^{(a)}_m\right)^2 - \prod_{b\sim a} Q^{(b)}_m}{Q^{(a)}_{m-1}}$$

$m\backslash a$	1	2	3	4
0	1	1	1	1
1	$Q^{(1)}_1$	$Q^{(2)}_1$	$Q^{(3)}_1$	$Q^{(4)}_1$
2	$Q^{(1)}_2$	$Q^{(2)}_2$	$Q^{(3)}_2$	$Q^{(4)}_2$
3	$Q^{(1)}_3$	$Q^{(2)}_3$	$Q^{(3)}_3$	$Q^{(4)}_3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

- initial question : Fix a positive integer $k \geq 1$ (level). How can we find $(Q^{(a)}_1)_{a\in I}$ such that $Q^{(a)}_k = 1$ for all $a \in I$?

level k restricted Q-system

Definition

For variables $\left(Q_m^{(a)}\right)$ with $0 \leq m \leq k$ and $a \in I$, consider a system of equations

$$\begin{cases} Q_0^{(a)} = 1 & a \in I \\ \left(Q_m^{(a)}\right)^2 = \prod_{b \sim a} \left(Q_m^{(b)}\right) + Q_{m-1}^{(a)} Q_{m+1}^{(a)} & 1 \leq m \leq k, a \in I \\ Q_{k+1}^{(a)} = 0 & a \in I \end{cases}$$

We call this system of equations the level k restricted Q-system.

- It is known that over $\mathbb{C}$ there exists a unique positive real solution of the level k restricted Q-system with $Q_k^{(a)} = 1$.

Rogers dilogarithm function

The Rogers dilogarithm function is defined by

$$L(x) = -\frac{1}{2}\int_0^x \frac{\log(1-y)}{y} + \frac{\log(y)}{1-y} dy$$

for $x \in (0,1)$. We set $L(0) = 0$ and $L(1) = \pi^2/6$ so that L is continuous on $[0,1]$.

- many functional identities are satisfied. For example,

$$L(x)+L(1-xy)+L(y)+L(\frac{1-y}{1-xy})+L\left(\frac{1-x}{1-xy}\right) = \frac{\pi^2}{2} = 3L(1)$$

- shows up in many places (e.g. number theory, algebraic K-theory, hyperbolic geometry)
- computes central charges for some conformal field theories

Dilogarithm identities for conformal field theories

Let $\left(Q_m^{(a)}\right)_{0\leq m\leq k, a\in I}$ be the unique positive real solution of the level k restricted Q-system and let

$$f_m^{(a)} := 1 - \frac{Q_{m-1}^{(a)} Q_{m+1}^{(a)}}{(Q_m^{(a)})^2} = \frac{\prod_{b\sim a} Q_m^{(b)}}{(Q_m^{(a)})^2}.$$

Theorem (Bazhanov,Kirillov,Reshetikhin '87,$\cdots$, Nakanishi '09, IIKKN '10)

Let $\mathfrak{g}$ be a simple Lie algebra of rank r (of types ADE). Then

$$\frac{6}{\pi^2}\sum_{a\in I}\sum_{m=1}^{k-1} L(f_m^{(a)}) = \frac{h(k-1)r}{h+k}$$

where h denote the Coxeter number of $\mathfrak{g}$.

- Originated in the Thermodynamic Bethe ansatz (TBA)

Kirillov-Reshetikhin (KR) modules

- Let q be a non-zero complex number which is not a root of unity
- KR modules form a special class of finite dimensional modules of the quantum affine algebra $U_q(\hat{\mathfrak{g}})$
- For given $\mathfrak{g}$, a KR module can be parametrized by $a \in I$, $m \in \mathbb{Z}_{\geq 0}$, and $u \in \mathbb{C}^\times$ (spectral parameter) and is denoted by $W_m^{(a)}(u)$.

Kirillov-Reshetikhin (KR) modules

- The quantized universal enveloping algebra $U_q(\mathfrak{g})$ is contained in $U_q(\hat{\mathfrak{g}})$ as a subalgebra
- From $U_q(\hat{\mathfrak{g}})$-module $W_m^{(a)}(u)$, we can get a finite dimensional $U_q(\mathfrak{g})$-module $\mathrm{res}\, W_m^{(a)}(u)$ by restriction and then u can be ignored

Theorem (Nakajima '03, Hernandez '06)

Let $Q_m^{(a)}$ be the character of $\mathrm{res}\, W_m^{(a)}(u)$. Then $\{Q_m^{(a)} | a \in I, m \in \mathbb{Z}_{\geq 0}\}$ satisfy the unrestricted Q-system (In fact, their q-characters satisfy the T-system).

character solutions of the unrestricted Q-systems

For $X = A_r$, we have $Q_m^{(a)} = \chi_{m\omega_a}$ for all $a \in I$ and $m \in \mathbb{Z}_{\geq 0}$
For $X = D_r$,

$$Q_m^{(a)} = \begin{cases} \sum \chi_{k_a\omega_a + k_{a-2}\omega_{a-2} + \cdots + k_1\omega_1} & 1 \leq a < r-1, a \equiv 1 \pmod 2, \\ \sum \chi_{k_a\omega_a + k_{a-2}\omega_{a-2} + \cdots + k_0\omega_0} & 1 \leq a < r-1, a \equiv 0 \pmod 2, \\ \chi_{m\omega_a} & a = r-1, r \end{cases}$$

where $\omega_0 = 0$ and the summation is over all nonnegative integers satisfying $k_a + k_{a-2} + \cdots + k_1 = m$ for a odd and $k_a + k_{a-2} + \cdots + k_0 = m$ for a even.
All known for classical types and partially known for exceptional types

using characters to solve the level k restricted Q-system

- We can get a solution of unrestricted Q-system over $\mathbb{C}$ by specializing characters $Q_m^{(a)}$ at elements of $\mathfrak{h}$ or $\mathfrak{h}^*$
- For a given level $k \geq 1$, which elements give rise to solutions of level k restricted Q-systems?
- Among those elements which one satisfies $Q_m^{(a)} > 0$ for all $0 \leq m \leq k$ and $a \in I$?

a short sentence from an old paper

In general one can show that $f_r^{(K)}\left(\frac{\pi}{r+g}\right) - 1$, $1 \leqslant K \leqslant rg(\mathfrak{g})$ always.
to the following (conjectural) identity

$$\sum_{K=1}^{rg(\mathfrak{g})} \sum_{m=1}^{r-1} L\left(f_m^{(K)}\left(\frac{\pi}{r+g}\right)\right) = \left\{\frac{r \cdot \dim \mathfrak{g}}{r+g} - rg(\mathfrak{g})\right\} \cdot \frac{\pi^2}{6}.$$

A. N. Kirillov (1989), Identities for the Rogers Dilogarithm Function Connected with Simple Lie Algebras. Journal of Soviet Mathematics 47 (translated from a paper in 1987)

Kuniba-Nakanishi-Suzuki (KNS) conjecture

Let $\rho = \sum_{i=1}^{r} \omega_i \in P$ the Weyl vector.

Conjecture (Kirillov '87, Kuniba-Nakanishi-Suzuki '92)

Let $\mathcal{D}_m^{(a)} = Q_m^{(a)}(\frac{\rho}{h+k})$ for each (a, m). For each $a \in I$, it satisfies the following properties :

- *(positivity) $\mathcal{D}_m^{(a)} > 0$ for $0 \leq m \leq k$.*
- *(unit boundary condition) $\mathcal{D}_k^{(a)} = 1$.*
- *(occurrence of 0) $\mathcal{D}_{k+1}^{(a)} = \mathcal{D}_{k+2}^{(a)} = \cdots = \mathcal{D}_{k+h-1}^{(a)} = 0$.*
- *(symmetry) $\mathcal{D}_m^{(a)} = \mathcal{D}_{k-m}^{(a)}$ for $1 \leq m \leq k-1$.*
- *(unimodality) $\mathcal{D}_{m-1}^{(a)} < \mathcal{D}_m^{(a)}$ for $1 \leq m \leq \lfloor \frac{k}{2} \rfloor$ where $\lfloor x \rfloor$ is the floor function.*

- It had been known to be true for $X = A_r$
- Now proven for all classical types and type E_6

quantum dimension

Note that the character χ_λ evaluated at $\frac{\rho}{h+k}$ is given by the Weyl formula

$$\mathcal{D}_{\hat{\lambda}} := \chi_\lambda\left(\frac{\rho}{h+k}\right) = \frac{\prod_{\alpha>0} \sin \frac{\pi(\lambda+\rho|\alpha)}{h+k}}{\prod_{\alpha>0} \sin \frac{\pi(\rho|\alpha)}{h+k}}$$

where $(\cdot|\cdot)$ is the standard bilinear form on P such that $(\theta|\theta) = 2$.

- We call it the quantum dimension
- It is not necessarily positive! (this is the primary source of headaches)

numerical example : $X = D_5, k = 4$

$$\begin{bmatrix} 1.000 & 1.000 & 1.000 & 1.000 & 1.000 \\ 3.732 & 8.464 & 14.93 & 4.732 & 4.732 \\ 5.464 & 15.93 & 33.32 & 7.464 & 7.464 \\ 3.732 & 8.464 & 14.93 & 4.732 & 4.732 \\ 1.000 & 1.000 & 1.000 & 1.000 & 1.000 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1.000 & 1.000 & 1.000 & 1.000 & 1.000 \\ 3.732 & 8.464 & 14.93 & 4.732 & 4.732 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

another numerical example : $X = D_5, k = 4$

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1.414 & 2. & 1.414 & 0.9239 - 0.3827i & 0.9239 + 0.3827i \\
0 & 2. & 0 & -0.7071 - 0.7071i & -0.7071 + 0.7071i \\
-1.414 & 2. & -1.414 & -0.3827 + 0.9239i & -0.3827 - 0.9239i \\
-1 & 1 & -1 & i & -i \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-1 & 1 & -1 & i & -i \\
-1.414 & 2. & -1.414 & 0.3827 + 0.9239i & 0.3827 - 0.9239i \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}$$

summary so far

In order to find solutions of level k restricted Q-systems over $\mathbb{C}$, study the image of the KR-modules under the homomorphism

$$\mathrm{Rep}(U_q(\hat{\mathfrak{g}})) \xrightarrow{\text{res}} \mathrm{Rep}(U_q(\mathfrak{g})) \xrightarrow{\mathrm{qdim}} \mathbb{C}$$

from finite to affine by adding an extra vertex

For a given classical weight $\lambda = \sum_{i=1}^{r} \lambda_i \omega_i \in P$, define its level k affinization $\hat{\lambda} = \sum_{i=0}^{r} \lambda_i \hat{\omega}_i \in \hat{P}^k$ where

- $\hat{P} = \mathbb{Z}\hat{\omega}_0 \oplus \mathbb{Z}\hat{\omega}_1 \oplus \cdots \oplus \mathbb{Z}\hat{\omega}_r$ the affine weight lattice
- $\hat{P}^k = \{\sum_{i=0}^{r} \lambda_i \hat{\omega}_i \in \hat{P} | \sum_{i=0}^{r} c_i \lambda_i = k\}$ where $\theta = \sum_{i=1}^{r} c_i \alpha_i$ is the highest root and $c_0 = 1$.
- Let $\hat{P}_+^k := \{\sum_{i=0}^{r} \lambda_i \hat{\omega}_i \in \hat{P}^k | \lambda_i \geq 0\}$ (the set of affine weights corresponding to classical weights in the Weyl alcove)
- For $\hat{\lambda} \in \hat{P}_+^k$, $\mathcal{D}_{\hat{\lambda}} > 0$.

WZW fusion ring

The WZW fusion ring $\mathrm{Fus}_k(\mathfrak{g})$ is a free $\mathbb{Z}$-module equipped with the basis $\{V_{\hat{\omega}} | \hat{\omega} \in \hat{P}^k_+\}$ and the product is given by

$$V_{\hat{\lambda}} \cdot V_{\hat{\mu}} = \sum_{\hat{\nu} \in \hat{P}^k_+} N^{\hat{\nu}}_{\hat{\lambda}\hat{\mu}} V_{\hat{\nu}}$$

where the fusion coefficient $N^{\hat{\nu}}_{\hat{\lambda}\hat{\mu}}$ can be computed by the Verlinde formula

$$N^{\hat{\nu}}_{\hat{\lambda}\hat{\mu}} = \sum_{\hat{\omega} \in \hat{P}^k_+} \frac{S_{\hat{\lambda},\hat{\omega}} S_{\hat{\mu},\hat{\omega}} \overline{S_{\hat{\nu},\hat{\omega}}}}{S_{\hat{0},\hat{\omega}}}.$$

Here $S_{\hat{\lambda},\hat{\mu}} = C \sum_{w \in W} (-1)^{\ell(w)} \exp\left(-\frac{2\pi i (w(\lambda+\rho)|\mu+\rho)}{k+h}\right)$ and C is a certain normalizing factor depending only on $\mathfrak{g}$ and k.

- commutative and associative as a ring and $V_{k\hat{\omega}_0}$ is the unity
- there exists an involution $V^*_{\hat{\omega}} := V_{\hat{\omega}^*}$

affine Weyl group

- The affine Weyl group $\hat{W}$ is generated by $s_0, s_1, \cdots, s_r$ and acts on $\hat{P}$ by

$$s_i\hat{\omega}_j = \hat{\omega}_j - \delta_{ij}\hat{\alpha}_i$$

- For a given $\hat{\mu} \in \hat{P}^k$, there exists a unique element $\hat{\lambda} \in \hat{P}^k$ such that $w(\hat{\mu} + \hat{\rho}) = \hat{\lambda} + \hat{\rho}$ for some $w \in \hat{W}$, $(\lambda + \rho|\alpha_a) \geq 0, \forall a \in I$ and $(\lambda + \rho|\theta) \leq k + h^\vee\}$.

another description of the WZW fusion ring

Let us define $\beta_k : \mathrm{Rep}(\mathfrak{g}) \to \mathrm{Fus}_k(\mathfrak{g})$ by

$$\beta_k(V_\lambda) := \begin{cases} 0 & \text{if } \mathcal{D}_{\hat{\lambda}} = 0 \\ (-1)^{\ell(w)} V_{\hat{\lambda}'} & \text{if } \mathcal{D}_{\hat{\lambda}} \neq 0 \end{cases} \tag{2.1}$$

where $\hat{\lambda}' \in \hat{P}_+^k$ is the unique element $\hat{\lambda}' \in \hat{P}_+^k$ such that

$$\hat{\lambda}' = w \cdot \hat{\lambda} := w(\hat{\lambda} + \hat{\rho}) - \hat{\rho} \tag{2.2}$$

for some $w \in \hat{W}$.
In short, move the weight in the Weyl chamber into the Weyl alcove by using the affine Weyl group.
This surjective map induces an isomorphism

$$\beta_k : \mathrm{Rep}(\mathfrak{g}) / \ker \beta_k \cong \mathrm{Fus}_k(\mathfrak{g})$$

and $\ker \beta_k$ is called the fusion ideal.

there are 4 good rings here, not just 3

In order to understand the KNS conjecture, intead of studying

$$\mathrm{Rep}(U_q(\hat{\mathfrak{g}})) \xrightarrow{\mathrm{res}} \mathrm{Rep}(U_q(\mathfrak{g})) \xrightarrow{\mathrm{qdim}} \mathbb{C},$$

look at the image of the KR-modules under the homomorphism

$$\mathrm{Rep}(U_q(\hat{\mathfrak{g}})) \xrightarrow{\mathrm{res}} \mathrm{Rep}(U_q(\mathfrak{g})) \xrightarrow{\beta_k} \mathrm{Fus}_k(\mathfrak{g}) \quad (\xrightarrow{\mathrm{qdim}} \mathbb{C})$$

example : $X = A_3, k = 3$

This is easy since the decomposition is simple.

$$\begin{bmatrix} W_0^{(1)} & W_0^{(2)} & W_0^{(3)} \\ W_1^{(1)} & W_1^{(2)} & W_1^{(3)} \\ W_2^{(1)} & W_2^{(2)} & W_2^{(3)} \\ W_3^{(1)} & W_3^{(2)} & W_3^{(3)} \\ W_4^{(1)} & W_4^{(2)} & W_4^{(3)} \\ W_5^{(1)} & W_5^{(2)} & W_5^{(3)} \\ W_6^{(1)} & W_6^{(2)} & W_6^{(3)} \\ W_7^{(1)} & W_7^{(2)} & W_7^{(3)} \\ \vdots & \vdots & \vdots \end{bmatrix} \xrightarrow{\beta_k \circ \mathrm{res}} \begin{bmatrix} V_{3\hat{\omega}_0} & V_{3\hat{\omega}_0} & V_{3\hat{\omega}_0} \\ V_{2\hat{\omega}_0+\hat{\omega}_1} & V_{2\hat{\omega}_0+\hat{\omega}_2} & V_{2\hat{\omega}_0+\hat{\omega}_3} \\ V_{\hat{\omega}_0+2\hat{\omega}_1} & V_{\hat{\omega}_0+2\hat{\omega}_2} & V_{\hat{\omega}_0+2\hat{\omega}_3} \\ V_{3\hat{\omega}_1} & V_{3\hat{\omega}_2} & V_{3\hat{\omega}_3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -V_{3\hat{\omega}_1} & V_{3\hat{\omega}_2} & -V_{3\hat{\omega}_3} \\ \vdots & \vdots & \vdots \end{bmatrix}$$

example : $X = D_5, k = 4$

It's easy for $a = 1, 4, 5$:

$$\begin{bmatrix} W_0^{(1)} & W_0^{(4)} & W_0^{(5)} \\ W_1^{(1)} & W_1^{(4)} & W_1^{(5)} \\ W_2^{(1)} & W_2^{(4)} & W_2^{(5)} \\ W_3^{(1)} & W_3^{(4)} & W_3^{(5)} \\ W_4^{(1)} & W_4^{(4)} & W_4^{(5)} \\ W_5^{(1)} & W_5^{(4)} & W_5^{(5)} \\ W_6^{(1)} & W_6^{(4)} & W_6^{(5)} \\ W_7^{(1)} & W_7^{(4)} & W_7^{(5)} \\ W_8^{(1)} & W_8^{(4)} & W_8^{(5)} \\ W_9^{(1)} & W_9^{(4)} & W_9^{(5)} \\ W_{10}^{(1)} & W_{10}^{(4)} & W_{10}^{(5)} \\ W_{11}^{(1)} & W_{11}^{(4)} & W_{11}^{(5)} \\ W_{12}^{(1)} & W_{12}^{(4)} & W_{12}^{(5)} \end{bmatrix} \xrightarrow{\beta_k \circ \mathrm{res}} \begin{bmatrix} V_{4\hat{\omega}_0} & V_{4\hat{\omega}_0} & V_{4\hat{\omega}_0} \\ V_{3\hat{\omega}_0+\hat{\omega}_1} & V_{3\hat{\omega}_0+\hat{\omega}_4} & V_{3\hat{\omega}_0+\hat{\omega}_5} \\ V_{2\hat{\omega}_0+2\hat{\omega}_1} & V_{2\hat{\omega}_0+2\hat{\omega}_4} & V_{2\hat{\omega}_0+2\hat{\omega}_5} \\ V_{\hat{\omega}_0+3\hat{\omega}_1} & V_{\hat{\omega}_0+3\hat{\omega}_4} & V_{\hat{\omega}_0+3\hat{\omega}_5} \\ V_{4\hat{\omega}_1} & V_{4\hat{\omega}_4} & V_{4\hat{\omega}_5} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ V_{4\hat{\omega}_1} & V_{4\hat{\omega}_4} & V_{4\hat{\omega}_5} \end{bmatrix}.$$

example : $X = D_5, k = 4$

For $a = 2$ and $0 \leq m \leq 4$,

$$\begin{bmatrix} W_0^{(2)} \\ W_1^{(2)} \\ W_2^{(2)} \\ W_3^{(2)} \\ W_4^{(2)} \end{bmatrix} \xrightarrow{\text{res}} \begin{bmatrix} V_{4\hat{\omega}_0} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} + V_{2\hat{\omega}_2} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} + V_{2\hat{\omega}_2} + V_{3\hat{\omega}_2-2\hat{\omega}_0} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} + V_{2\hat{\omega}_2} + V_{-2\hat{\omega}_0+3\hat{\omega}_2} + V_{-4\hat{\omega}_0+4\hat{\omega}_2} \end{bmatrix}$$

Let us compute

$$\beta_k(\text{res } W_4^{(2)}) = \beta_k(V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} + V_{2\hat{\omega}_2} + V_{-2\hat{\omega}_0+3\hat{\omega}_2} + V_{-4\hat{\omega}_0+4\hat{\omega}_2}).$$

The shifted action of the affine Weyl group gives

$$s_0 \cdot (-2\hat{\omega}_0 + 3\hat{\omega}_2) = 2\hat{\omega}_2,$$
$$s_0 \cdot (-4\hat{\omega}_0 + 4\hat{\omega}_2) = 2\hat{\omega}_0 + \hat{\omega}_2.$$

Thus $\beta_k(\text{res } W_4^{(2)}) = V_{4\hat{\omega}_0}$.

example : $X = D_5, k = 4$

Thus, for $a = 2$ and $0 \leq m \leq 4$,

$$\begin{bmatrix} W_0^{(2)} \\ W_1^{(2)} \\ W_2^{(2)} \\ W_3^{(2)} \\ W_4^{(2)} \end{bmatrix} \xrightarrow{\text{res}} \begin{bmatrix} V_{4\hat{\omega}_0} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} + V_{2\hat{\omega}_2} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} + V_{2\hat{\omega}_2} + V_{3\hat{\omega}_2-2\hat{\omega}_0} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} + V_{2\hat{\omega}_2} + V_{-2\hat{\omega}_0+3\hat{\omega}_2} + V_{-4\hat{\omega}_0+4\hat{\omega}_2} \end{bmatrix}$$

$$\xrightarrow{\beta_k} \begin{bmatrix} V_{4\hat{\omega}_0} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_2} + V_{2\hat{\omega}_0+\hat{\omega}_2} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} \\ V_{4\hat{\omega}_0} \end{bmatrix}$$

example : $X = D_5, k = 4$

For $a = 2, 3$ and $0 \leq m \leq h + k = 12$,

$$\begin{bmatrix} W_0^{(2)} & W_0^{(3)} \\ W_1^{(2)} & W_1^{(3)} \\ W_2^{(2)} & W_2^{(3)} \\ W_3^{(2)} & W_3^{(3)} \\ W_4^{(2)} & W_4^{(3)} \\ W_5^{(2)} & W_5^{(3)} \\ W_6^{(2)} & W_6^{(3)} \\ W_7^{(2)} & W_7^{(3)} \\ W_8^{(2)} & W_8^{(3)} \\ W_9^{(2)} & W_9^{(3)} \\ W_{10}^{(2)} & W_{10}^{(3)} \\ W_{11}^{(2)} & W_{11}^{(3)} \\ W_{12}^{(2)} & W_{12}^{(3)} \end{bmatrix} \mapsto \begin{bmatrix} V_{4\hat{\omega}_0} & V_{4\hat{\omega}_0} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} & V_{3\hat{\omega}_0+\hat{\omega}_1} + V_{2\hat{\omega}_0+\hat{\omega}_3} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_2} + V_{2\hat{\omega}_0+\hat{\omega}_2} & V_{2\hat{\omega}_0+2\hat{\omega}_1} + V_{2\hat{\omega}_3} + V_{\hat{\omega}_0+\hat{\omega}_1+\hat{\omega}_3} \\ V_{4\hat{\omega}_0} + V_{2\hat{\omega}_0+\hat{\omega}_2} & V_{\hat{\omega}_0+3\hat{\omega}_1} + V_{2\hat{\omega}_1+\hat{\omega}_3} \\ V_{4\hat{\omega}_0} & V_{4\hat{\omega}_1} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ V_{4\hat{\omega}_0} & V_{4\hat{\omega}_1} \end{bmatrix}.$$

boundary of Q-system

Lemma

Let $(\tau_a)_{a\in I}$ *be as follows :*

	τ_a
A_r	$\hat{\omega}_a$
D_r	$\begin{cases} \hat{\omega}_1 & \textit{if } 1 \leq a \leq r-2 \textit{ and } a \equiv 1 \pmod 2 \\ \hat{\omega}_0 & \textit{if } 1 \leq a \leq r-2 \textit{ and } a \equiv 0 \pmod 2 \\ \hat{\omega}_a & \textit{if } a = r-1 \textit{ or } a = r \end{cases}$
$\vdots$	$\vdots$

Then $(V_{k\tau_a})_{a\in I}$ *satisfies the system of equations*

$$\left(Q^{(a)}\right)^2 = \prod_{b\sim a} Q^{(b)}, \quad a \in I.$$

main theorem : lifting of the KNS conjecture

Theorem (L '13)

For each $a \in I$, let us define $R_m^{(a)}$ in the fusion ring by

$$R_m^{(a)} = \begin{cases} \beta_k(\mathrm{res}\, W_m^{(a)}) & 0 \le m \le \lfloor \frac{k}{2} \rfloor \\ V_{k\tau_a}\left(\beta_k(\mathrm{res}\, W_{k-m}^{(a)})\right)^* & \lfloor \frac{k+1}{2} \rfloor \le m \le k \end{cases}$$

and $R_{-1}^{(a)} = R_{k+1}^{(a)} = 0$. Then $\left(R_m^{(a)}\right)$ is a positive solution of the level k restricted Q-system in the fusion ring.

- proved for all classical types.
- the quantum dimension version of the KNS conjecture follows as a corollary (positivity, symmetry, unit boundary condition,...)

new characterization of the positive solution

Corollary

For each $a \in I$ and $0 \le m \le k$, the positive number $\mathcal{D}_m^{(a)} = \mathrm{qdim}\, W_m^{(a)}$ is the Perron-Frobenius eigenvalue of the fusion matrix (which is non-negative integral) associated to $R_m^{(a)}$.

problems : positivity and periodicity

Conjecture

For $a \in I$, let τ_a as before and $\sigma_a = e^{-2\pi i(\tau_a|\rho)}$. The following properties hold :

1. $\beta_k(\mathrm{res}\, W_m^{(a)})$ *is positive for* $0 \leq m \leq k$,
2. $\beta_k(\mathrm{res}\, W_{k-m}^{(a)}) = V_{k\tau_a}\left(\beta_k(\mathrm{res}\, W_m^{(a)})\right)^*$ *for* $0 \leq m \leq k$,
3. $\beta_k(\mathrm{res}\, W_k^{(a)}) = V_{k\tau_a}$,
4. $\beta_k(\mathrm{res}\, W_{k+1}^{(a)}) = \beta_k(\mathrm{res}\, W_{k+2}^{(a)}) = \cdots = \beta_k(\mathrm{res}\, W_{(k+h)-1}^{(a)}) = 0$,
5. $\beta_k(\mathrm{res}\, W_{m+n(k+h)}^{(a)}) = \sigma_a^n V_{k\tau_a}^n \beta_k(\mathrm{res}\, W_m^{(a)})$ *for* $0 \leq m \leq (k+h) - 1$ *and* $n \in \mathbb{Z}_{\geq 0}$.

completely verified only in type A and up to $0 \leq m \leq \lfloor \frac{k+1}{2} \rfloor$ for all classical types and some special $a \in I$'s

problems : periodicity and linear recurrence in Q-systems

Let $Q_m^{(a)} = \chi(\mathrm{res}\, W_m^{(a)})$ be the classical character and let $q_a := Q_1^{(a)}$ so that $Q_m^{(a)} \in \mathbb{Z}[q_1, \cdots, q_r]$

Conjecture

For each $a \in I$, there exists a positive integer ℓ_a and polynomials $C_k^{(a)} \in \mathbb{Z}[q_1, \cdots, q_r]$, $k = 0, \cdots, \ell_a$ such that the following holds

$$\sum_{k=0}^{\ell_a} (-1)^k C_k^{(a)} Q_{n-k}^{(a)} = 0$$

for all $n \in \mathbb{Z}$. Here $C_0^{(a)} = C_{\ell_a}^{(a)} = 1$.

work in progress ($C_k^{(a)}$ is sometimes related to an exterior power of certain KR module)

problems : $k = \infty$ analogue of the KNS conjecture

Conjecture

The dimension $\left(\dim W_m^{(a)}\right)$ *is the unique specialization* $Q_m^{(a)}$ *of* $\chi(\operatorname{res} W_m^{(a)})$ *in* $\mathbb{R}$ *satisfying the following properties :*

- *(positivity)* $Q_m^{(a)} > 0$ *for each* $m \in \mathbb{Z}_{\geq 0}$ *and* $a \in I$*,*
- *(polynomial growth) The sequence* $\left(Q_m^{(a)}\right)_{m \in \mathbb{Z}_{\geq 0}}$ *is of polynomial growth for each* $a \in I$

It implies that one can come up with the dimensions of KR modules by using Q-systems only without any knowledge in representation theory

problems

- What can we say about the map

$$\mathrm{Rep}(U_q(\hat{\mathfrak{g}})) \to \mathrm{Rep}(U_q(\mathfrak{g})) \to \mathrm{Fus}_k(\mathfrak{g})$$

  if $q = e^{\pi i/t(k+h^\vee)}$

- combinatorial understanding of the coefficients in

$$\beta_k(\mathrm{res}\, W_m^{(a)}) = \sum_{\hat{\lambda}\in \hat{P}_+^k} Z(a,m,\hat{\lambda}) V_{\hat{\lambda}} \in \mathrm{Fus}_k(\mathfrak{g})$$

  (fermionic formula? and what if we include finer statistics like the energy on KR crystals)

- Thank you!